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A characterization of strategic complementarities

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Abstract

I characterize games for which there is an order on strategies such that the game has strategic complementarities. I prove that, with some qualifications, games with a unique equilibrium have complementarities if and only if Cournot best-response dynamics has no cycles; and that all games with multiple equilibria have complementarities.

As applications of my results, I show: 1. That generic $2X2$ games either have no pure-strategy equilibria, or have complementarities. 2. That generic two-player finite ordinal potential games have complementarities.

Resumen

Caracterizo los juegos para los que hay un orden de las estrategias que hacen que el juego tenga complementariedades estratégicas. Demuestro que los juegos con un equilibrio único tienen complementariedades si y sólo si la dinámica Cournot de mejor respuesta no tiene ciclos; y que todos los juegos con equilibrios múltiples tienen complementariedades.

Como aplicaciones de mis resultados obtengo que: 1. Genéricamente los juegos $2X2$ tienen complementariedades o no tienen equilibrios en estrategias puras. 2. Genéricamente, los juegos finitos de dos jugadores con potencial ordinal, tienen complementariedades.

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1 Introduction

A game has strategic complementarities (Topkis 1979, Vives 1990) if, given an order on players' strategies, an increase in one player's strategy makes the other players want to increase their strategies. For example, the players could be firms in price competition; if each firm's optimal price is an increasing function of the prices set by their opponents, the game has strategic complementarities. In games of strategic complementarities (GSC), Nash equilibria have a certain order structure; in particular, there is a smallest and largest equilibrium (Topkis, Vives, Zhou (1994)). Further, the set of all rationalizable strategies, and the set of limits of adaptive learning, is bounded below by the smallest equilibrium and above by the largest equilibrium (Vives 1990, Milgrom and Roberts 1990, Milgrom and Shannon 1994). GSC are a well-behaved class of games, and a useful tool for economists. In this paper I basically characterize the full set of GSC, and argue that complementarities is a useful assumption only when coupled with additional structure, not by itself.

We normally introduce strategic complementarities by assuming supermodular—or quasisupermodular—payoffs. The crucial feature for most results on GSC is that the game's best-response correspondence is monotone increasing, and the assumption of supermodular payoffs is sufficient for monotone increasing best-responses. I shall argue that it is enough, for the purpose of this paper, to define a GSC as a game for which there is a partial order on strategies so that best-responses are monotone increasing (and such that strategies have a lattice structure).

Consider the coordination game in Figure 1, is it a GSC? That is, is there an order on player's strategies so that best-responses are monotone increasing? Yes, let α be smaller than β. Then, player 2's best response to α is α and to β is β. So, when 1 increases her strategy from α to β, 2's best response increases from α to β. Similarly for 1. So, with this simple order the coordination game is a GSC.

The order on strategies that makes the coordination game a GSC is artificial. This is always the case: how strategies are ordered is never part of the description of a game. We as analysts use the order as a tool, therefore we are justified in choosing the order to conform to our theory. In fact, typically results about GSC are used in applications by introducing an order on strategies that was not present in the problem that motivated the model.

	α	β
α	$2,2$	$0,0$
β	$0,0$	$1,1$

Coordination

	H	T
H	$1,-1$	$-1,1$
T	$-1,1$	$1,-1$

Matching Pennies

Figure 1: 2x2 games, GSC?

Now consider the game "Matching Pennies" in Figure 1, is this game a GSC? Let H be smaller than T, then 2's best response to H, T, is larger than 2's best response to T, H. The same problem shows up if we set T smaller than H. So, we need to order the player's strategies in different ways. Say that for player 1 H is smaller than T, but that for player 2 T is smaller than H. Now 2's best-responses are increasing, but 1's best response is not—when 2 increases her strategy from T to H, 1's best response decreases from T to H. So matching pennies is not a GSC.

What does this depend on? Why is the simple coordination game a GSC but not matching pennies? My results are:

- With some qualifications, a game with a unique pure-strategy Nash equilibrium is a GSC if and only if Cournot best-response dynamics have no cycles except for the equilibrium.
- A game with two or more pure-strategy Nash equilibria is always a GSC.

I develop two applications from these results:

- Generically, $2X2$ games are either GSC or have no pure-strategy equilibria.
- Generically, a finite two-player ordinal potential game is a GSC; and ordinal potential games with more than two players need not be GSC

I now discuss my main results.

With some qualifications, in games with a unique Nash equilibrium, strategic complementarities is equivalent to the absence of cycles in Cournot best-response dynamics. If b is the game's best-response function (the product of the players' best-response functions) then Cournot best-response dynamics starting at x is defined by

$x_0 = x$, $x_n = b(x_{n-1})$, $n = 1, 2, \ldots$. Absence of cycles means that, if x is not an equilibrium, Cournot best-response dynamics starting at x never returns to x, i.e. $x_n \neq x$ for all $n \geq 1$, or, equivalently, that $x \neq b^n(x)$ for all $n \geq 1$. In finite games, absence of cycles is equivalent to global stability, that is that $b^n(x)$ converges to the equilibrium for all x. In infinite games, absence of cycles is a weaker condition than global stability—so I show that a game with a unique, globally stable, equilibrium is a GSC, but the converse is not true.

A game with two or more equilibria is a GSC, so the vast majority of games that we encounter in applied work are GSC. The order in the coordination game of Figure 1 that makes its best-response function increasing involves making one Nash equilibrium the smallest point in the joint strategy space, and the other equilibrium the largest point in the strategy space. I show that, if a game has at least two equilibria, the same trick always works; we can order the strategies such that one equilibrium is the largest strategy profile and the other equilibrium is the smallest, and such that best-responses are monotone increasing. This result has implications for the use of complementarities to obtain predictions in games.

The literature on GSC has developed a set-valued prediction concept: the set of strategies that are larger than the smallest Nash equilibrium and smaller than the largest Nash equilibrium. This "interval prediction" contains all rationalizable strategies, and all strategies that are limits of adaptive learning. Is the interval prediction in general a sharp prediction? Milgrom and Roberts suggest that the answer may be negative:

> Indeed, for some games, these bounds are so wide that our result is of little help: it is even possible that these bounds are so wide that the minimum and maximum elements of the strategy space are equilibria. (Milgrom and Roberts 1990, p. 1258)

Milgrom and Roberts go on to argue that, in some models, "the bounds are quite narrow." They present as examples an arms-race game, and a class of Bertrand oligopoly models, where there is a unique equilibrium.

My results imply that this is *generally* the case: if a game does not have a unique equilibrium, the interval prediction is essentially vacuous, as all games with multiple equilibria are GSC where the smallest and largest equilibria are the smallest and largest strategy profiles. The explanation for this is that games with multiple equilibria always involve a kind of coordination problem, namely the problem of coordinating on one equilibrium. This coordination problem can be formalized through an order on strategies that makes the game a GSC.

I show that, in some games, no order that makes the game a GSC avoids a trivial interval prediction. In fact, the coordination game in Figure 1 is such a game. On the other hand, in the coordination game, all strategy profiles are rationalizable, which may suggest that the interval prediction coincides in general with rationalizability. I present an example where no order that makes the game a GSC, avoids strictly dominated strategies in the interval prediction—this is further evidence that the interval prediction is problematic.

I wish to emphasize that GSC are still a very useful tool. In a given game, complementarities jointly with the rest of the structure of the game can provide sharp results. In a sense, complementarities is like compactness—most spaces we work with can be compactified, but compactness is still a very powerful tool. The point of this paper is that complementarities alone do not possess important predictive power, but that we can use complementarities to understand properties of a particular game. For example, Topkis's (1979) algorithm for finding the smallest and largest Nash equilibria in GSC is very useful on a given game with complementarities; and once Topkis's algorithm has delivered an interval prediction we can judge if it is sharp or not *in that particular game*. Other examples are Amir's (1996) elegant methods for analyzing Cournot oligopoly models using complementarities, and the equilibrium uniqueness results in the literature on global games (Morris and Shin 2000). [1]

Lippman, Mamer, and McCardle (1987), Sobel (1988), Milgrom and Roberts (1990), Milgrom and Shannon (1994), Milgrom and Roberts (1994), and Echenique (2001a) present comparative statics results for a parameterized GSC. They prove

[1] Amir's analysis does not proceed by introducing a convenient order on strategy spaces like I do (see also Amir and Lambson (2000)). My results—which, unlike Amir's, require prior knowledge of some equilibria of the game—imply that there are more Cournot oligopoly models that can be made into a GSC than the ones identified by Amir. But the structure, in addition to complementarities, that Amir uses in his paper enables him to obtain substantive results about Cournot oligopoly, and his results cannot be generalized using my characterization.

that, if a parameter is complementary to players' choices, some selections of equilibria are monotone increasing in the parameter. My results prompt the question: can any systematic comparative statics conclusion be rationalized as coming from a parameterized GSC? I show that this is not the case: there are parameterized models (with multiple equilibria) such that no order delivers the comparative statics as an application of results for GSC. In this sense, comparative statics acts as an "identifying restriction." Comparative statics imply that we do not have enough degrees of freedom in selecting the order on strategies so as to make the game a GSC and, at the same time, preserve monotone comparative statics.

In the usual definition of GSC—supermodular games and games with ordinal complementarities—there is a link between the order and the topology on the strategy spaces that, among other things, ensures the existence of equilibria (by Tarski's Theorem), and that if equilibrium is unique it must be globally stable. Here I construct orders that make best-responses monotone increasing, but that do not have any relation to the topology on the strategy spaces. In finite games this does not matter: a finite game without equilibria cannot be a GSC, and a game with a unique equilibrium is a GSC if and only if this equilibrium is globally stable. In infinite games, though, there may be GSC without equilibria, and GSC with a unique equilibrium that is not globally stable. I wish to emphasize that, despite this technical problem, for games with multiple equilibria, all of Vives's, Topkis's and Milgrom and Roberts's results hold trivially with the order that I construct.

In Section 2 I give some preliminary definitions. Section 3 presents my results for games with a unique equilibrium, and section 4 presents my results for games with multiple equilibria. Comparative statics restrictions are presented in section 5. In section 6 I discuss an application to $2X2$ games, and in section 7 an application to potential games. Section 8 contains the proof of Theorem 3.

2 Preliminaries

A detailed discussion of the concepts defined in this subsection can be found in Topkis (1998). A pair $(X, \leq)$, where X is a set and $\leq$ is a transitive, reflexive, antisymmetric binary relation, is a ***partially ordered*** set; $(X, \leq)$ is ***totally ordered*** if, for all

$x, y \in X$, $x \leq y$ or $y \leq x$ ($\leq$ is then a total order on X); $(X, \leq)$ is a ***lattice*** if whenever $x, y \in X$, both $x \wedge y = \inf\{x, y\}$ and $x \vee y = \sup\{x, y\}$ exist in X. Note that a totally ordered set is a lattice.

A lattice $(X, \leq)$ is ***complete*** if for every nonempty subset A of X, $\inf A, \sup A$ exist in X. A nonempty subset A of X is a ***sublattice*** if for all $x, y \in A$, $x \wedge_X y, x \vee_X y \in A$, where $x \wedge_X y$ and $x \vee_X y$ are obtained taking the infimum and supremum as elements of X (as opposed to using the relative order on A). A nonempty subset $A \subseteq X$ is ***subcomplete*** if $B \subseteq A$, $B \neq \emptyset$ implies $\inf_X B, \sup_X B \in A$, again taking inf and sup of B as a subset of X. For two subsets A, B of X, say that A is smaller than B in the ***strong set order*** if $a \in A$, $b \in B$ implies $a \wedge b \in A, a \vee b \in B$.

Let $(X, \leq)$ be a lattice. Say that a correspondence $\phi : X \twoheadrightarrow X$ is ***weakly increasing*** over $A \subseteq X$ if $x, y \in A$ and $x \leq y$ implies that there is $z \in \phi(x)$ and $z' \in \phi(y)$ with $z \leq z'$. Also, say that ϕ is ***increasing in the strong set order*** if $x < y$ implies that $\phi(x)$ is smaller in the strong set order than $\phi(y)$. Note that when ϕ is a function, i.e. single valued, both concepts coincide with the usual notion of "monotone weakly increasing."

A correspondence $\phi : X \twoheadrightarrow X$ takes ***finite values*** if $\phi(x)$ is a finite set for all $x \in X$.

3 Unique Equilibrium

I shall present all my results as results about functions, or correspondences, on a set X. The reader should think of the best-response function, or correspondence, of a game as the main application of my results. Let $\Gamma = \left\{I, \{u_i\}_{i \in I} \{S_i\}_{i \in I}\right\}$ be a normal-form game. That is, I is a set of players, and each player $i \in I$ is endowed with a strategy space S_i and a payoff function $u_i : X = \times_{j \in I} S_j \to \mathbf{R}$. If $\beta_i : X \twoheadrightarrow S_i$ is player i's best-response correspondence, $\beta_i(s) = \mathrm{argmax}_{\tilde{s}_i} u_i(\tilde{s}_i, s_{-i})$, then $\phi : X \twoheadrightarrow X$ defined by $\phi = \times_{j \in I} \beta_i$ is the game's best-response correspondence. If best-responses are always unique, then β_i is a function, and $f = \times_{j \in I} \beta_i$ is the game's best-response function. The set of fixed points of ϕ—or, if best-responses are unique, of f—coincides with the set of pure-strategy Nash equilibria of Γ. A game $\Gamma = \left\{I, \{u_i\}_{i \in I} \{S_i\}_{i \in I}\right\}$ is a ***game of strategic complementarities*** (GSC) if there is an order $\leq$ on X such that $(X, \leq)$ is a lattice and ϕ is weakly increasing. If Γ has unique best-responses, Γ is a GSC if there is an order $\leq$ on X such that $(X, \leq)$ is a lattice and f is a monotone increasing function.

The problem, then, is when can we find an order $\leq$ such that $(X, \leq)$ is a lattice and ϕ—or f—is weakly increasing?

Definition 1 *Let X be a set and $\phi : X \twoheadrightarrow X$ be a correspondence. A point $x \in X$ is a* ***cycle*** *of ϕ if there is $n \in \mathbf{N}$ such that $x \in \phi^n(x)$. Let X be a topological space and $\phi : X \twoheadrightarrow X$ a correspondence. A fixed point e is* ***globally stable*** *for ϕ if, for every $x \in X$ and every sequence $\{x_k\}$ with $x = x_0$ and $x_k \in \phi(x_{k-1})$, $x_k \to e$. These two definitions are extended to functions $f : X \to X$ by interpreting f as a correspondence with singleton values.*

If players meet to play Γ over and over again, and if they, in each round of play n, choose a best response to their opponents' play in $n-1$, we say that they follow Cournot best-response dynamics. If ϕ is Γ's best-response correspondence, all Cournot best-response dynamics are generated by $x_n \in \phi(x_{n-1})$ (for some initial x_0). Then, absence of cycles means that no Cournot best-response dynamics will get caught in a cycle. And global stability means that players that engage in Cournot best-response dynamics will eventually approach Nash equilibrium play. [2]

The existence of an order such that a function, or a correspondence, is increasing is a non-topological statement, so it cannot depend on global stability. The absence of cycles is a non-topological condition which is weaker than global stability: global stability implies the absence of cycles other than e, but absence of cycles does not imply global stability. For example, let X be the unit disk in $\mathbf{R}^2$ and f be a rotation of X by an irrational number, then f has a unique fixed point, $(0, 0)$, and no cycles, but $(0, 0)$ is clearly not globally stable.

Theorem 2 *Let X be a set and $f : X \to X$ be a function on X that has a unique fixed point $e \in X$. There is a total order $\leq$ on X such that f is monotone increasing if and only if f has no cycles besides e.*

[2] Cournot best-response dynamics is a very naive learning model, my results do not depend of any virtues of Cournot dynamics as a learning model. The absence of cycles should be viewed as a technical condition. It is true, though, that Cournot dynamics is easy to implement on a computer, and therefore the condition of absence of cycles is easy to verify computationally (in a finite game).

	α	β	γ
α	$2,2$	$0,0$	$0,0$
β	$0,0$	$1,-1$	$-1,1$
γ	$0,0$	$-1,1$	$1,-1$

Γ_1

	α	β
α	$1,-1$	$2,-2$
β	$0,0$	$1,-1$

Γ_2

Figure 2: A non-GSC, and a GSC that is zero-sum.

Theorem 2 may have some mathematical interest, independently of the application to game theory that I emphasize. The proof of Theorem 2 follows from Theorem 3 below.

Let $\Gamma = \left\{I, \{u_i\}_{i\in I} \{S_i\}_{i\in I}\right\}$ be a normal-form game with a unique Nash equilibrium. If all players have unique best-responses—which, as these are pure-strategy best-responses, is generically the case when Γ is a finite game—then, by Theorem 2, Γ is a GSC if and only if Cournot best-response dynamics has no cycles except for e. [3] In particular, if e is globally stable under Cournot best-response dynamics, then Γ is a GSC. One implication of these results is that a game that satisfies the dominant diagonal condition in Gabay and Moulin (1980), and that has thus a unique, globally stable, equilibrium, is a GSC.

As an illustration of Theorem 2, consider Γ_1, the $3X3$ game on the left in Figure 2. This game has a unique pure-strategy equilibrium, and best-response dynamics has the cycles $(\alpha,\alpha) \to (\alpha,\alpha)$, $(\beta,\alpha) \to (\alpha,\gamma) \to (\gamma,\alpha) \to (\alpha,\beta) \to (\beta,\alpha)$, and $(\beta,\beta) \to (\beta,\gamma) \to (\gamma,\gamma) \to (\gamma,\beta) \to (\beta,\beta)$. Therefore there is no order on strategies such that the game in Figure 2 is a GSC.

Consider Γ_2, the zero-sum game on the right in Figure 2 as a second illustration of Theorem 2. This is a GSC because it has a unique equilibrium and no cycles. The intuitive idea that zero-sum games and GSC are different is therefore false.

3.1 Non-unique best-responses.

Theorem 2 for functions can be generalized to correspondences with finite values, but the characterization is not completely tight. Absence of cycles implies the existence of a total order that makes the correspondence weakly monotone, but a weakly monotone

[3] See section 3.2 for the difference between lattice and totally ordered strategy spaces.

correspondence may have cycles, as weak monotonicity does not control all selections from the correspondence.

Theorem 3 *Let $\phi : X \twoheadrightarrow X$ be a correspondence with a unique fixed point $e \in X$. If ϕ takes finite values and has no cycles besides e, there is a total order $\leq$ on X such that ϕ is weakly increasing. Further, if $\{e\} = \phi(e)$ and there is a total order $\leq$ on X such that ϕ is increasing in the strong set order, then ϕ has no cycles besides e.*

The proof of Theorem 3 is in section 8.

I need the hypothesis of a correspondence with finite values to use non-standard analysis the way I do. If Γ is a game with unique best-responses, of course the best-responses have finite values, and both weakly increasing and increasing in the strong set order coincide with monotone increasing, so Theorem 3 implies Theorem 2. Also, if Γ is a finite game, then best-responses take finite values. I have not been able to extend the characterization in Theorem 3 to arbitrary games with infinite strategy spaces and non-unique best-responses.

Global stability implies absence of cycles, thus we obtain

Corollary 4 *Let X be a topological space and $\phi : X \twoheadrightarrow X$ a correspondence with finite values and a unique fixed point $e \in X$. If e is globally stable then there is a total order $\leq$ on X such that ϕ is weakly increasing.*

3.2 Totally ordered strategy space vs. a lattice strategy space

Theorems 2 and 3 give necessary and sufficient conditions for a game to be a GSC with a totally ordered strategy space. I defined GSC as games with lattice strategy spaces, is there some loss in focusing on totally ordered strategies? Yes, but it is essentially a technical problem: Consider Theorem 2. If a game's best-response function does not have cycles, then the game is a GSC because there is a total order on strategies such that best responses are monotone increasing, and a totally ordered set is a lattice. On the other hand, if Γ is a GSC where each S_i is a complete lattice, and best-responses f are continuous, then a unique Nash equilibrium is globally stable (Vives 1990, Milgrom and Roberts 1990) so f has no cycles and there is a total order on strategies such that f is increasing. In particular this implies that, in finite games,

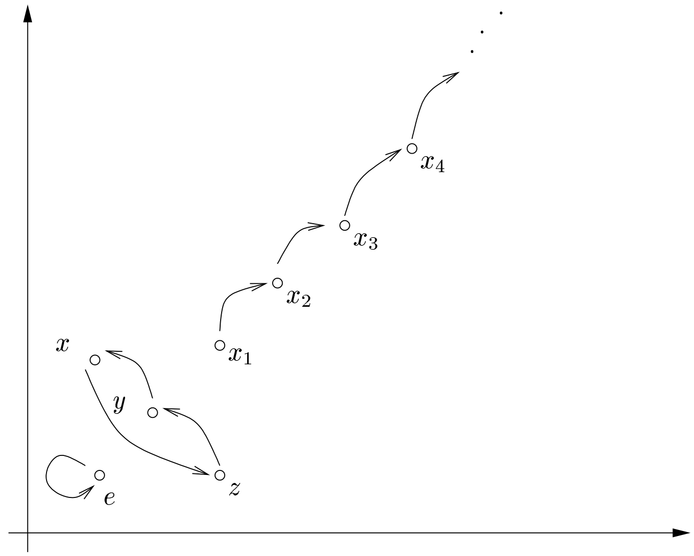

Figure 3: Arrows show action of $f : \{e, x, y, z, x_1, x_2, \ldots\} \to \{e, x, y, z, x_1, x_2, \ldots\}$

a game with a unique equilibrium is a GSC if and only if best responses have no cycles.

In GSC where strategy spaces are non-complete lattices, best-responses may have cycles. Consider the example in Figure 3. The arrows in the figure show the action of a function $f : \{e, x, y, z, x_1, x_2, \ldots\} \to \{e, x, y, z, x_1, x_2, \ldots\}$. Let the infinite set $\{e, x, y, z, x_1, x_2, \ldots\}$ be ordered as a subset of $\mathbf{R}^2$. The function f has a unique fixed point, e, a cycle $z \to y \to x \to z$, and it is monotone increasing. The example works because Cournot best-response dynamics starting at x_1 (or any other point larger than x, y and z) is a monotone increasing sequence $x_1, x_2, \ldots$ that does not converge to a fixed point different from e. So the characterization in Theorem 2 is not completely tight. As I remarked earlier, complementarities is a non-topological condition, so the characterization must be independent of completeness on strategy spaces. In a sense, then, it is impossible to avoid problems like the one in Figure 3.

4 Multiple Equilibria

A game with two or more Nash equilibria is easily transformed into a GSC. It is enough to set one equilibrium as the smallest strategy profile, and the other as the largest. I argue below that it is in general not possible to improve on this result.

4.1 Games with Multiple Equilibria are GSC

Theorem 5 *Let $\phi : X \twoheadrightarrow X$ be a correspondence. If ϕ has at least two different fixed points, then there is an order $\leq$ on X such that $(X, \leq)$ is a complete lattice, ϕ is weakly increasing, and the set of fixed points of ϕ is a complete sublattice of X.*

Proof: Let $\underline{e}$, $\overline{e}$ be different fixed points of ϕ. Define $\leq$ on X by $x \leq y$ if and only if one of the following is true: $x = y$, $x = \underline{e}$, or $y = \overline{e}$. Then, for any non-singleton $A \subseteq X$, $\underline{e}$ is the unique lower bound on A, hence $\inf A = \underline{e}$; similarly $\sup A = \overline{e}$ as $\overline{e}$ is the unique upper bound on A. Thus, for all $x, y \in X$, $x \vee y, x \wedge y \in X$, and for all non-empty $A \subseteq X$, $\inf A$, $\sup A \in X$, so X is a complete lattice.

Now, let $x \leq y$ and $x \neq y$; if $x = \underline{e}$ then $\underline{e} \in \phi(x)$, so $\underline{e} \leq z$ for all $z \in \phi(y)$, and similarly if $y = \overline{e}$. If $x = y$ then any $z = z' \in \phi(x) = \phi(y)$ satisfies $z \leq z'$. Thus if $x \leq y$ there is $z \in \phi(x)$ and $z' \in \phi(y)$ with $z \leq z'$, so ϕ is weakly increasing.

Finally, if A is a non-singleton set of fixed points, then $\underline{e}$ is the unique lower bound, and $\overline{e}$ is the unique upper bound, on A. So, $\inf A = \underline{e}$ and $\sup A = \overline{e}$. But $\underline{e}$ and $\overline{e}$ are fixed points, so the set of fixed points is a complete sublattice of X. ■

Remarks:

1. The usual definitions of GSC ensure that best-responses are increasing in the strong set order, and that ϕ takes subcomplete- sublattice-values. The order that I construct in Theorem 5 does not guarantee that ϕ has these properties (unless, of course, ϕ is a function), but Topkis's, Vives's, Milgrom and Roberts's, and Milgrom and Shannon's results are (trivially) true with the constructed order.

2. By Zhou's version of Tarski's Fixed Point Theorem the equilibrium set of a GSC is a complete lattice. Topkis, Vives and Zhou present examples where it

is not a sublattice (see Echenique (2001b), though). By Theorem 5 there is a partial order on X such that this equilibrium set is a complete sublattice, without eliminating complementarities.

For clarity, I include the statement of Theorem 5 when ϕ is a function as:

Corollary 6 *Let $f : X \to X$ be a function. If f has at least two different fixed points, then there is an order $\leq$ on X such that $(X, \leq)$ is a complete lattice, f is monotone increasing, and the set of fixed points of f is a complete sublattice of X.*

Theorem 5 implies that the smallest and largest equilibrium are, respectively, the smallest and largest element in the strategy space. We can improve on Theorem 5 in a class of finite games, but we need a definition first. Let X be a finite set, and $f : X \to X$ be a function. If $e \in X$ is a fixed point of f, the ***basin*** of e, denoted $\mathbb{B}_e$, is the set of points $x \in X$ such that $e = f^n(x)$ for some n.

Theorem 7 *Let X be a finite set, and $f : X \to X$ be a function. Let $\underline{e}, \overline{e} \in X$ be two fixed points of f. There is an order $\leq$ on X such that $(X, \leq)$ is a complete lattice, f is monotone increasing, and, for all $z \in \mathbb{B}_{\underline{e}}$ and $z' \in \mathbb{B}_{\overline{e}}$, $z < \underline{e}$ and $\overline{e} < z'$. So, the interval prediction is*

$$[\,\underline{e}, \overline{e}\,] = X \backslash (\mathbb{B}_{\underline{e}} \cup \mathbb{B}_{\overline{e}}) \cup \{\underline{e}, \overline{e}\}\,.$$

Proof: Let g be the restriction of f to $\mathbb{B}_{\underline{e}}$. The range of g is in $\mathbb{B}_{\underline{e}}$, so $g : \mathbb{B}_{\underline{e}} \to \mathbb{B}_{\underline{e}}$. Since $g^n(x) = \underline{e}$ for some n for all x in $\mathbb{B}_{\underline{e}}$, g has exactly one fixed point, $\underline{e}$, and no cycles but $\underline{e}$. Then, by Theorem 2 there is a total order $\preceq$ on $\mathbb{B}_{\underline{e}}$ such that g is monotone increasing. Also, the proof of Theorem 3 (step 2) shows that $\underline{e}$ is the largest element in $\mathbb{B}_{\underline{e}}$. Similarly there is a total order $\preceq'$ on $\mathbb{B}_{\overline{e}}$ such that the restriction of f to $\mathbb{B}_{\overline{e}}$ is monotone increasing and $\overline{e}$ is the smallest element in $\mathbb{B}_{\overline{e}}$.

Now, let $h : X \backslash (\mathbb{B}_{\underline{e}} \cup \mathbb{B}_{\overline{e}}) \cup \{\underline{e}, \overline{e}\} \to X \backslash (\mathbb{B}_{\underline{e}} \cup \mathbb{B}_{\overline{e}}) \cup \{\underline{e}, \overline{e}\}$ be the restriction of f to $X \backslash (\mathbb{B}_{\underline{e}} \cup \mathbb{B}_{\overline{e}}) \cup \{\underline{e}, \overline{e}\}$. By Theorem 5 there is an order $\preceq''$ on $\hat{X} = X \backslash (\mathbb{B}_{\underline{e}} \cup \mathbb{B}_{\overline{e}}) \cup \{\underline{e}, \overline{e}\}$ such that $(\hat{X}, \preceq'')$ is a complete lattice, h is monotone increasing, $\underline{e}$ is the smallest element, and $\overline{e}$ is the largest element.

Finally, define the order $\leq$ on X by $x \leq y$ if and only if one of the following is true, a) $x \in \mathbb{B}_{\underline{e}}, y \notin \mathbb{B}_{\underline{e}}$ b) $x \notin \mathbb{B}_{\underline{e}} \cup \mathbb{B}_{\overline{e}}, y \in \mathbb{B}_{\underline{e}}$ c) $x, y \in \mathbb{B}_{\underline{e}}$ and $x \preceq y$ d) $x, y \in \mathbb{B}_{\overline{e}}$ and $x \preceq' y$, or e) $x, y \notin \mathbb{B}_{\underline{e}} \cup \mathbb{B}_{\overline{e}}$ and $x \preceq'' y$. It is routine to check that $(X, \leq)$ is a complete lattice, and that f is monotone increasing. ■

4.2 Discussion of the interval prediction.

In a game with multiple equilibria, the order constructed in Theorem 5 implies that the interval prediction concept in, among others, Milgrom and Roberts (1990) is trivial. Is there some order that also makes games with multiple equilibria GSC, and for which the interval prediction is non-trivial? That is, is it possible to improve on Theorem 5 so that interval predictions are sharper? I will argue that the answer is, in general, no. There are examples, like the coordination game in the Introduction, where the only order that makes a game a GSC involves a trivial interval prediction.

In the coordination game, all strategy profiles are rationalizable. This may suggest that the interval prediction coincides in general with rationalizability. In fact, Theorem 7 leaves the non-equilibrium elements in the basins of two equilibria out of the interval prediction, and the non-equilibrium element in the basin of an equilibrium are not rationalizable. I present an example where no order that makes the game a GSC, avoids strictly dominated strategies in the interval prediction.

Coordination games are discussed in items 1 and 2. Dominated strategies in the interval prediction is discussed in 3.

1. Consider the coordination game in Figure 1. I shall show that there is no order on strategies that preserves complementarities, and where the interval prediction is sharper than the whole strategy space. Let

$$b = b_1 \times b_2 : \{\alpha, \beta\}^2 \to \{\alpha, \beta\}^2$$

be the game's best-response function, and $\leq$ be an order on $\{\alpha, \beta\}^2$ such that b is monotone increasing. Now, it must be that (α, β) and (β, α) are incomparable under $\leq$. To see this, let $(\alpha, \beta) < (\beta, \alpha)$, then $b(\beta, \alpha) = (\alpha, \beta) < b(\alpha, \beta) = (\beta, \alpha)$, and b is not increasing. Similarly if $(\beta, \alpha) < (\alpha, \beta)$. Then, for $(\{\alpha, \beta\}^2, \leq)$ to be a lattice, it must be that $(\alpha, \beta) \vee (\beta, \alpha)$ and $(\alpha, \beta) \wedge (\beta, \alpha)$ equals either (α, α) and (β, β) or, respectively, (β, β) and (α, α). But $(\alpha, \beta) \vee (\beta, \alpha) \neq (\alpha, \beta) \wedge (\beta, \alpha)$, hence either (α, α) is the smallest strategy profile and (β, β) is the largest, or vice versa.

2. I show that this does not depend on the coordination game being $2X2$, it depends on the cycles in best-responses between the non-equilibrium strategy profiles (α, β) and (β, α). [4] In fact, consider the $4X4$ coordination game Γ_3 in Figure 4. In

[4] In both coordination games, all strategy profiles are rationalizable. So, it follows from Milgrom

	α	β	γ	δ
α	2, 2	0, 0	0, 0	0, 0
β	0, 0	0, 1	1, 0	0, 0
γ	0, 0	1, 0	0, 1	0, 0
δ	0, 0	0, 0	0, 0	1, 1

Γ_3

	α'	β'
α	−1, 0	−1, 1
β	2, 2	0, 0
γ	0, 0	2, 2

Γ_4

Figure 4:

this game, best-responses have three cyclical orbits (besides the fixed-point cycles $(\alpha,\alpha) \to (\alpha,\alpha)$ and $(\delta,\delta) \to (\delta,\delta)$):

$$\begin{aligned}
&(\beta,\alpha) \to (\alpha,\beta) \to (\gamma,\alpha) \to (\alpha,\gamma) \to (\beta,\alpha),\\
&(\delta,\gamma) \to (\beta,\delta) \to (\delta,\beta) \to (\gamma,\delta) \to (\delta,\gamma),\\
&\text{and } (\beta,\beta) \to (\gamma,\beta) \to (\gamma,\gamma) \to (\beta,\gamma) \to (\beta,\beta).
\end{aligned}$$

Then, if a partial order $\leq$ makes the $4X4$ coordination game into a GSC, no strategy that belongs to a cyclical orbit can be the smallest, or the largest, strategy profile of the game. To see this, suppose, for example, that (β,α) is the smallest strategy profile. Then $(\beta,\alpha) < (\alpha,\beta)$, so if b is monotone increasing, $b^3(\beta,\alpha) \leq b^3(\alpha,\beta)$. But $(\beta,\alpha) = b^3(\alpha,\beta)$ and $(\alpha,\beta) = b^3(\beta,\alpha)$. So it must be that the smallest and largest element in the strategy space is (α,α) and (δ,δ), as a finite lattice has a smallest and a largest element.

The coordination game shows that it is *in general* impossible to sharpen the Milgrom-Roberts interval predictions. Theorem 7 does not improve on the order from Theorem 5 in the $2X2$ and $4X4$ coordination games discussed above. The reason is that the basins of both extremal equilibria in these examples are singletons, and all Theorem 7 guarantees is that non-equilibrium elements in the basins of the two extremal equilibria are not in the interval prediction.

3. Consider Γ_4, the example on the right in Figure 4. In Γ_4, player 1 has a strictly dominated strategy, α, but there is no order on strategies that preserves complementarities, and such that the interval prediction rules out 1 playing α.

There are two pure-strategy equilibria in Γ_4, (γ,β') and (β,α'). Let us set $\underline{e} =$

and Roberts's results that all strategy profiles are in the interval prediction. I show this directly here and in item 1 to make explicit the role of cycles.

(γ, β') and $\overline{e} = (\beta, \alpha')$. It is easy to check that the basins of $\underline{e}$ and $\overline{e}$ are, respectively, $\mathbb{B}_{\underline{e}} = \{(\alpha, \beta'), \underline{e}\}$ and $\mathbb{B}_{\overline{e}} = \{\overline{e}\}$. Then, Theorem 7 implies that there is an order such that best-responses in Γ_4 are monotone increasing, and the interval prediction is

$$[\, \underline{e}, \overline{e}\,] = \{(\gamma, \beta'), (\alpha, \alpha'), (\beta, \beta'), (\gamma, \alpha'), (\beta, \alpha')\}\,.$$

Note that the strategy pair (α, α'), where 1 selects a strictly dominated strategy, is in the interval prediction.

In fact, it is unavoidable that (α, α') is in the interval prediction. Let $\leq$ be an order on $\{\alpha, \beta, \gamma\} \times \{\alpha', \beta'\}$ such that $(\{\alpha, \beta, \gamma\} \times \{\alpha', \beta'\}, \leq)$ is a complete lattice, and the best-response function, f, is monotone increasing. Say that $\underline{e} = (\gamma, \beta') < (\beta, \alpha') = \overline{e}$. Suppose we want $(\alpha, \alpha') \notin [\, \underline{e}, \overline{e}\,]$. Then, as (β, β') and (γ, α') are pairs of rationalizable strategies, we must have $(\beta, \beta'), (\gamma, \alpha') \in [\, \underline{e}, \overline{e}\,]$ (one can prove this directly, as in items 1 and 2 above). If $(\alpha, \alpha') < \underline{e}$ then $(\beta, \beta') = f(\alpha, \alpha') \leq \underline{e} = f(\underline{e})$, impossible since $(\beta, \beta') \in [\, \underline{e}, \overline{e}\,]$. Similarly, we cannot have $\overline{e} < (\alpha, \alpha')$, so (α, α') must be incomparable to either $\underline{e}$ or $\overline{e}$. Say it is incomparable to $\underline{e}$, then we must have $(\alpha, \beta') = (\alpha, \alpha') \wedge \underline{e}$. But then there are only two candidates for $(\alpha, \alpha') \vee \underline{e}$: (β, β') and (γ, α'). But (β, β') and (γ, α') are cycles, as $(\gamma, \alpha') = f(\beta, \beta')$ and $(\beta, \beta') = f(\gamma, \alpha')$. Then they cannot be ordered, and we cannot have a smallest upper bound on (α, α') and $\underline{e}$. Similarly, (α, α') and $\overline{e}$ cannot be unordered. So, it must be that $(\alpha, \alpha') \in [\, \underline{e}, \overline{e}\,]$.

5 Comparative Statics

In a parameterized GSC, if a parameter t is complementary to players' choices, there are selections of equilibria that are monotone increasing in t (Lippman, Mamer, and McCardle 1987, Sobel 1988, Villas-Boas 1997, Milgrom and Roberts 1990, Milgrom and Roberts 1994, Milgrom and Shannon 1994, Echenique 2001a). By theorems 3 and 5, most games can be rationalized as GSC. A natural question is: can any comparative statics conclusion be rationalized as coming from a parameterized GSC? The answer is no, comparative statics conclusions act as an "identifying condition" that restricts the choice of an order on strategy spaces.

Using the framework in Milgrom and Roberts (1994), the question can be phrased

	α	β
α	a, b	c, d
β	e, f	g, h

Γ_4

	α	β
α	$1, 1$	$0, 1$
β	$0, 1$	$1, 0$

Γ_5

Figure 5: Γ_3 used in the proof of, and Γ_4 illustrates, Proposition 8

as follows. Let $f_t, f_{t'} : X \to X$, let e be a fixed point of f_t, and e' a fixed point of $f_{t'}$. Is there an order on X such that $e < e'$ and such that $f_t(x) \leq f_{t'}(x)$ for all x?

In general the answer is no. Consider the following example, let $x, e, e' \in X$ with $e = f_t(e)$, $e' = f_{t'}(e')$ and $e \neq e'$. Suppose that $e' = f_t(x)$ and $e = f_{t'}(x)$. If there is an order $\leq$ such that $f_t(z) \leq f_{t'}(z)$ for all $z \in X$, then $e = f_t(e) \leq f_{t'}(e) = e'$ and $e < e'$ because $e \neq e'$. But $e' = f_t(x) \leq f_{t'}(x) = e$, a contradiction.

The example above shows that we cannot rationalize comparative statics conclusions by a model where the parameter is complementary to players' choices. Alternatively, is there an order on X such that $e < e'$ and such that f_t and $f_{t'}$ are monotone increasing functions? That is, can we rationalize a comparative statics conclusion as coming from parameterized GSC, even though the parameter may not be complementary to the strategies? The following example shows that this need not be the case. Let $x, e, e' \in X$ with $e = f_t(e)$, $e' = f_{t'}(e')$ and $e \neq e'$. Suppose that $x = f_t(e')$, $e = f_{t'}(x)$ and $e' = f_{t'}(e)$. Then $e < e'$ implies that $f_t(e) \leq f_t(e')$, so $e < x$. Then $f_{t'}(e) \leq f_{t'}(x)$ implies that $e' < e$, a contradiction.

6 Application 1: $2X2$ Games

As an application of Theorems 3 and 5 I show that, generically, $2X2$ games either have no equilibria or have complementarities. Essentially, then, $2X2$ games are either isomorphic to "Matching Pennies", or they are GSC.

Proposition 8 *Generically, a $2X2$ game either has no pure-strategy Nash equilibrium, or it is a GSC.*

Proof: Consider the game in Figure 5 on the left. Suppose that a, b, c, etc. are such that (β, β) is the unique Nash equilibrium, and (β, β) is a strict equilibrium, so.

$c < g$ and $f < h$. I will rule out all possible cycles in best-response dynamics, except for the cycle involving (β, β). Then Theorem 2 implies that the game is a GSC.

Since $c < g$ and $f < h$, (α, α) is not a best response to (α, β) or to (β, α). This rules out the cycles $(\alpha, \alpha) \to (\alpha, \beta) \to (\alpha, \alpha)$, $(\alpha, \beta) \to (\alpha, \alpha) \to (\alpha, \beta)$, $(\alpha, \alpha) \to (\beta, \alpha) \to (\alpha, \alpha)$, $(\beta, \alpha) \to (\alpha, \alpha) \to (\beta, \alpha)$, $(\alpha, \alpha) \to (\beta, \alpha) \to (\alpha, \beta) \to (\alpha, \alpha)$, and $(\alpha, \alpha) \to (\alpha, \beta) \to (\beta, \alpha) \to (\alpha, \alpha)$.

Finally, the cycle $(\beta, \alpha) \to (\alpha, \beta) \to (\beta, \alpha)$ requires that $a \geq e$ and $b \geq d$. But then (α, α) would be an equilibrium, contradicting that (β, β) is the unique Nash equilibrium. Similarly for $(\alpha, \beta) \to (\beta, \alpha) \to (\alpha, \beta)$. These are all possible cycles.

We have shown that all $2X2$ games with a unique, strict Nash equilibrium are GSC. All games with more than one equilibrium are GSC by Theorem 5. The property that a unique equilibrium is strict is generic in the class of $2X2$ games. ■

Proposition 8 does not extend to more complex games than $2X2$, see the example in Figure 2 on the left. The "genericity" qualification in Proposition 8 is necessary. Consider the game on the right in Figure 5. This game has a unique Nash equilibrium (α, α) in pure strategies, and a cycle, $(\alpha, \alpha) \to (\alpha, \beta) \to (\beta, \alpha) \to (\alpha, \alpha)$. The cycle is produced by player 2's indifference when 1 chooses α, which is non-generic.

7 Application 2: Ordinal Potential Games

A game $\Gamma = \left\{I, \{u_i\}_{i \in I} \{S_i\}_{i \in I}\right\}$ is an ***ordinal potential game*** if there is a function $P : S = \times_{i \in I} S_i \to \mathbf{R}$ such that

$$u_i(s_i, s_{-i}) < u_i(s_i', s_{-i}) \text{ if and only if } P(s_i, s_{-i}) < P(s_i', s_{-i}),$$

for all $s_i, s_i' \in S_i$, $s_{-i} \in S_{-i}$, and $i \in I$. Potential games were studied in detail by Monderer and Shapley (1996) (see their paper for references to earlier work on potential games).

My results shed some light on the relation between GSC and ordinal potential games. I show that, generically, a finite two-player ordinal potential game is a GSC; and that ordinal potential games with more than two players need not be GSC. [5]

[5] Further, it is easy to see that GSC need not be ordinal potential games. I do not discuss this here, the counterexamples are very simple.

	α	β
α	$-1,-1,-1$	$-1,-1,-1$
β	$1,1,0$	$0,0,1$
γ	$0,0,1$	$1,1,0$

α

	α	β
α	$1,1,1$	$1,0,1$
β	$0,1,1$	$0,0,0$
γ	$0,0,0$	$0,1,1$

β

Figure 6: A three-player ordinal potential game that is not a GSC.

Proposition 9 *Let Γ be a finite two-player game with unique best responses. If Γ is an ordinal potential game, then it is a GSC.*

Proposition 9 does not extend to games with more than two players. For a counterexample, consider the three-player game in Figure 6. Here, player 1 chooses a strategy in $S_1 = \{\alpha, \beta, \gamma\}$ (rows), player 2 chooses a strategy in $S_2 = \{\alpha, \beta\}$ (columns), and player 3 a strategy in $S_3 = \{\alpha, \beta\}$ (matrices). The payoffs are indicated in the figure.

The game in Figure 6 has a unique equilibrium, (α, α, β), and the best-response cycle $(\beta, \beta, \alpha) \to (\gamma, \alpha, \alpha) \to (\beta, \beta, \alpha)$ By Theorem 2, then, there is no order on strategies so that his game is a GSC. But by Monderer and Shapley's results, the game has a generalized ordinal potential, as it has the finite improvement property (this is a bit cumbersome to check). Now just perturb the game a bit, it still will not be a GSC because the best-response cycle will persist, and it will be an ordinal potential game by Monderer and Shapley's Corollary 2.6.

Proof of Proposition 9 To prove Proposition 9, I need two definitions and two lemmas. Let $\Gamma = \{S_1, S_2, u_1, u_2\}$ be a two-player game with unique best-responses, and let $\beta(s_1, s_2) = (\beta_1(s_2), \beta_2(s_1))$ be Γ's best-response function—note the minor change in notation. A *best-response cycle* of Γ is a sequence $(s^0, s^1, \ldots s^n)$ in $S = S_1 \times S_2$, where $s^k = \beta(s^{k-1})$, $1 \leq k \leq n$, $s^0 = s^n$, and $s^{k-1} \neq s^k$, $1 \leq k \leq n$. The use of the term "cycle" here is inconsistent with the rest of the paper, but no confusion should arise because it will be clear, at all times in this section, that a cycle is the whole path $(s^0, s^1, \ldots s^n)$, not a point in the range of the path.

An *infinite improvement path* in S is a sequence $(w^0, w^1, \ldots)$ such that:

- $(w^0, w^1, \ldots)$ is not eventually constant (i.e. there is no K such that the sequence

$(w^K, w^{K+1}, \ldots)$ is constant).

- w^{k-1} and w^k differ in at most one component,
- if w^{k-1} and w^k differ, and i is the player that changes strategy between w^{k-1} and w^k, then $u_i(w^{k-1}) < u_i(w^k)$.

The definition of an infinite improvement path differs from the one in Monderer and Shapley. However, it is immediate to modify their Lemma 2.3 to show:

Lemma 10 *(Monderer and Shapley (1996)) An ordinal potential game cannot have an infinite improvement path.*

I do not include a proof of Lemma 10 in this paper.

Lemma 11 *Let Γ be a two-player game with unique best responses and a unique Nash equilibrium. If Γ has a best-response cycle, then it has an infinite improvement path.*

Proof: Let $(s^0, s^1, \ldots s^n)$ be a best-response cycle. We must have $n \geq 2$, or $(s^0, s^1, \ldots s^n)$ is a constant sequence.

I claim that either $s_1^0 \neq s_1^2$, $s_2^0 \neq s_2^2$, or both hold. Suppose, by way of contradiction, that $s_1^0 = s_1^2$ and $s_2^0 = s_2^2$. Then $s_1^0 = \beta_1(s_2^1)$ and $s_2^0 = \beta_2(s_1^1)$. Now $s_1^1 = \beta_1(s_2^0)$ and $s_2^1 = \beta_2(s_1^0)$ imply that (s_1^0, s_2^1) and (s_1^1, s_2^0) are Nash equilibria. But there is a unique Nash equilibrium, so $(s_1^0, s_2^1) = (s_1^1, s_2^0)$; impossible, as $(s_1^0, s_2^0) = s^0 \neq s^1 = (s_1^1, s_2^1)$. Suppose, without loss of generality, that $s_2^0 \neq s_2^2$.

Extend the sequence $(s^0, s^1, \ldots s^n)$ to the infinite sequence

$$(s^0, s^1, \ldots s^{n-1}, s^0, s^1, \ldots).$$

So, s_1^{n+1} refers to s_1^1, s_2^{2n+3} to s_2^3, and so on. Construct the sequence

$$w = (w^0, w^1, w^2, \ldots) = ((s_1^1, s_2^0), (s_1^1, s_2^2), (s_1^3, s_2^2), (s_1^3, s_2^4), (s_1^5, s_2^4), (s_1^5, s_2^6), \ldots).$$

I shall show that w is an infinite improvement path.

If n is odd, we can construct the following finite sequences:

$$\begin{array}{ll}
y^0 = (s_1^1, s_2^0) & z^0 = (s_1^0, s_2^1) \\
y^1 = (s_1^1, s_2^2) & z^1 = (s_1^2, s_2^1) \\
y^2 = (s_1^3, s_2^2) & z^2 = (s_1^2, s_2^3) \\
y^3 = (s_1^3, s_2^4) & z^3 = (s_1^4, s_2^3) \\
\vdots & \vdots \\
y^{n-1} = (s_1^n, s_2^{n-1}) = (s_1^0, s_2^{n-1}) & z^{n-1} = (s_1^{n-1}, s_2^n) = (s_1^{n-1}, s_2^0)
\end{array}$$

If n is even, we shall only need the y sequence as constructed, with the modification that we get $y^{n-1} = (s_1^{n-1}, s_2^n) = (s_1^{n-1}, s_2^0)$.

It is easy, if somewhat cumbersome, to show that: If n is odd, then

$$w = (y^0, y^1, \ldots y^{n-1}, z^0, z^1, \ldots z^{n-1}, y^0, y^1, \ldots).$$

And that, if n is even, then

$$w = (y^0, y^1, \ldots y^{n-1}, y^0, y^1, \ldots).$$

In both cases, w is not eventually constant, as $y_0 \neq y_1$ because $s_2^0 \neq s_2^2$.

Let $w^{k-1} \neq w^k$. If $w^{k-1} = (s_1^k, s_2^{k-1})$, and $w^k = (s_1^k, s_2^{k+1})$, then $s_2^{k-1} \neq s_2^{k+1} = \beta_2(s_1^k)$. Best-responses are unique, then $u_2(s_1^k, s_2^{k-1}) < u_2(s_1^k, s_2^{k+1})$. Similarly if $w^k = (s_1^{k+1}, s_2^k)$, and $w^{k-1} = (s_1^{k-1}, s_2^k)$. So, w is an infinite improvement path. ■

By Monderer and Shapley's (1996) Corollary 2.2, Γ has at least one equilibrium. If it has two or more equilibria, it is a GSC by Theorem 5. Let Γ have a unique equilibrium. Lemma 10 implies that Γ cannot have an infinite improvement path, so Lemma 11 implies that it cannot have a best-response cycle; by Theorem 2, then, Γ is a GSC. ■

8 Proof of Theorem 3

I prove the second statement in Step 1. Steps 2 and 3 prove the first statement when X is finite—Step 2 constructs an order such that ϕ is weakly increasing, and Step 3 checks that this is a total order. Steps 4 and 5 prove the first statement for arbitrary X by non-standard methods. The idea is to embed X in a hyperfinite set, apply the result for finite sets to get an order that works in the hyperfinite set, and then restrict the order to X.

STEP 1. Let $\leq$ be a total order on X such that ϕ is increasing in the strong set order, and let $x \neq e$. Suppose, by way of contradiction, that there is $\{x_m\}_{m=0}^K \subseteq X$ such that $x = x_0 = x_K$ and $x_m \in \phi(x_{m-1})$, $1 \leq m \leq K-1$. Note that we must have $x_m \neq e$ for all m because $\phi(e) = \{e\}$ would imply that $x = e$.

Since X is totally ordered by $\leq$, either $x < x_{K-1}$ or $x_{K-1} < x$, as $x = x_{K-1}$ is ruled out because x is not a fixed point. Suppose that $x < x_{K-1}$, I will show by induction that $x_K < x_{K-1} < \ldots < x_0 = x$, a contradiction. First, $x < x_{K-1}$ implies that $\phi(x) \ni x_1$ is smaller than $\phi(x_{K-1}) \ni x$ in the strong set order, so $x_1 \wedge x \in \phi(x)$. Now, $x_1 \wedge x \in \{x_1, x\}$ because $\leq$ is a total order, and $x \notin \phi(x)$ because x is not a fixed point. So we must have $x_1 < x$. Now for the inductive step, I want to show that, if $x_m < x_{m-1}$, then $x_{m+1} < x_m$ $(1 \leq m \leq K-1)$. If $x_m < x_{m-1}$, then $\phi(x_m) \ni x_{m+1}$ is smaller than $\phi(x_{m-1}) \ni x_m$ in the strong set order. Then $\{x_{m+1}, x_m\} \ni x_{m+1} \wedge x_m \in \phi(x_m)$, but $x_m \notin \phi(x_m)$ as $x_m \neq e$, so $x_{m+1} < x_m$.

If, instead, $x > x_{K-1}$ we can apply an analogous argument to reach a contradiction. So there cannot be a K with $x \in \phi^K(x)$.

STEP 2. Let X be finite. Let $\phi : X \twoheadrightarrow X$ be a correspondence with a unique fixed point $e \in X$, and no cycles besides e. For all $x \in X$ there is at least one sequence $\{x_m\}_{m=0}^K$ with $x_0 = x$, $x_m \in \phi(x_{m-1})$, $m = 1, 2, \ldots K$, and $x_K = e$; that is a *path* connecting x and e, K is the length of the path. To see that such a path must exist, note that, by absence of cycles, if $x_m \neq e$ for $m = 1, 2, \ldots K$, then $\{x_m\}_{m=0}^K$ are all distinct, so X has at least $K+1$ elements. X is finite, so we must have, for K large enough, that $x_K = e$. For each $x \in X$, let K_x be the smallest K such that there is a path of length K connecting x and e. Clearly, $K_e = 0$. Fix, for each x, a path of minimal length $\{x_m\}_{m=0}^{K_x}$ connecting x and e. Note that $K_{x_1} = K_x - 1$, as the K's are minimal.

Define $X_m = \{x \in X : K_x = m\}$, $m = 0, 1, \ldots M$, where M is such that $X = \cup_{m=0}^M X_m$. The collection $\{X_m\}_{m=0}^M$ is a partition of X because it covers X and has disjoint elements, as the K_xs are minimal.

I shall define recursively an order $\leq_m$ on X_m, $m = 0, 1, \ldots K$. Let $\preceq$ be a total order on X such that e is the largest element of X (for example, embed X in $\mathbf{N}$

such that e is mapped to a number larger than any other element in X, and take the relative order on X). Let $\leq_0$ on X_0, be the restriction of $\preceq$ to X_0. Given a total order $\leq_m$ on X_m, let $\leq_{m+1}$ on X_{m+1} be defined by, $x \leq_{m+1} y$ if $x_1 <_m y_1$ or if $x_1 = y_1$ and $x \preceq y$. Note that $\leq_m$ is in fact a total order because it is a lexicographic order.

Finally, define a total order $\leq$ on X by $x \leq y$ if either $k_y < k_x$ or if $k_y = k_x$ and $x \leq_{k_x} y$. Note that e is the largest element in X.

STEP 3. I shall check that ϕ is weakly increasing. Let $x, y \in X$ and $\{x_m\}_{m=0}^{K_x}$, $\{y_m\}_{m=0}^{K_y}$ be the paths of minimal length connecting x and y to e from step 2.

Let $x \leq y$. If $K_y < K_x$ then there is $x_1 \in \phi(x)$ and $y_1 \in \phi(y)$, $x_1 \in \{x_m\}_{m=0}^{K_x}$, $y_1 \in \{y_m\}_{m=0}^{K_y}$, with $K_{y_1} = K_y - 1 < K_x - 1 = K_{x_1}$ because the K's are minimal. Then, $x_1 \leq y_1$. If $K_y = K_x$ then there is $x_1 \in \phi(x)$ and $y_1 \in \phi(y)$, $x_1 \in \{x_m\}_{m=0}^{K_x}$, $y_1 \in \{y_m\}_{m=0}^{K_y}$ with $x_1 \leq_{K_x+1} y_1$. Then $K_{y_1} = K_y - 1 = K_x - 1 = K_{x_1}$, because the K's are minimal. Then, $x_1 \leq y_1$. In both cases there is $x_1 \in \phi(x)$ and $y_1 \in \phi(y)$ with $x_1 \leq y_1$, thus proving that ϕ is weakly increasing

To prove that $\leq$ is a total order, note first that $x \leq x$ because x is in some X_m, and $x \leq_m x$, as $\leq_m$ is an order; thus $\leq$ is reflexive. If $x \leq y$ and $y \leq x$ then we must have $K_x = K_y$, and then $x = y$ because $\leq_{K_x}$ is an order. So, $\leq$ is antisymmetric. Let $x \leq y$ and $y \leq z$. If $K_y < K_x$ or $K_z < K_y$ then $K_z < K_x$ and $x \leq z$. If $K_x = K_y = K_z$ then $x \leq z$ follows from transitivity of $\leq_{K_x}$. Thus $\leq$ is transitive. Finally, $\leq$ is total because if $x, y \in X$ then there is m, n such that $x \in X_m$ and $y \in X_n$. If $m < n$ or $n < m$, x and y are ordered. If $n = m$, x and y are ordered because $\leq_m$ is a total order.

STEP 4. Suppose first that $\phi(e) = \{e\}$. Let H be a hyperfinite set with $X \subseteq H \subseteq *X$. For all $x \in X$, $\phi(x)$ is finite, then $*\phi$ takes hyperfinite values, and therefore $*\phi(H)$ is hyperfinite, as it is the hyperfinite union of hyperfinite sets. Let $\tilde{H} = H \cup *\phi(H)$, $\tilde{H}$ is hyperfinite.

By the Transfer Principle, $h \in H$ implies $(h \in *\phi(h) \Rightarrow h = e)$. Define $\tilde{\phi} : \tilde{H} \to \tilde{H}$ by

$$\tilde{\phi}(h) = \begin{cases} *\phi(h) & \text{if } h \in H \\ \{e\} & \text{if } h \in \tilde{H} \backslash H. \end{cases}$$

$\tilde{\phi}$ is internal, and $\tilde{\phi}(e) = *\phi(e) = \{e\}$. I show that e is the only fixed point of $\tilde{\phi}$. Suppose $x \in \tilde{\phi}(x)$. First, if $x \in H$, then $\tilde{\phi}(x) = *\phi(x)$, so $x \in *\phi(x)$ implies that $x = e$ by Transfer. Second, if $x \in \tilde{H} \backslash H$ then $\tilde{\phi}(x) = \{e\}$, so $x = e \in H$, a contradiction.

To show that $\tilde{\phi}$ has no cycles except for e, suppose that $x \in \tilde{\phi}^K(x)$ for some $x \in \tilde{H}$ and $K \in *\mathbf{N}$. Note that $x \in H$ because $x \in \tilde{H} \backslash H$ implies that $\tilde{\phi}^m(x) = \{e\}$ for all m, and $e \in H$. There is a sequence $\{x_m\}_{m=0}^K$ with $x_0 = x_K = x$ and $x_m \in \tilde{\phi}(x_{m-1})$, $m = 1, 2, \ldots K$. If there is m such that $x_m \in \tilde{H} \backslash H$ then $e = x_l$, $m \leq l \leq K$, so $x = e$. On the other hand, if $x_{m-1} \in H$ for all m, then $x_m \in \tilde{\phi}(x_{m-1}) = *\phi(x_{m-1})$ for all m. Then x is a cycle of $*\phi$, so $x = e$ by Transfer.

Then, $\tilde{H}$ is hyperfinite, $\tilde{\phi} : \tilde{H} \twoheadrightarrow \tilde{H}$ has a unique fixed point e, and no cycles but e. By steps 2 and 3 above and the Transfer Principle, there is a total order $\lhd$ on H such that $x \lhd e$ for all $x \in H$, and such that $x \lhd y$ implies that there is $z \in \tilde{\phi}(x)$ and $z' \in \tilde{\phi}(y)$ with $z \lhd z'$. Let $\leq$ be the restriction of $\lhd$ to $X \subseteq \tilde{H}$, $\leq$ is a total order on X, and e is its largest element.

To verify that ϕ is weakly increasing, let $x \leq y$. Then, $x \lhd y$ implies that there is $z \in \tilde{\phi}(x)$ and $z' \in \tilde{\phi}(y)$ with $z \lhd z'$. But $\tilde{\phi}(x) = *\phi(x) = \phi(x)$ and $\tilde{\phi}(y) = *\phi(y) = \phi(y)$, as $x, y \in X \subseteq H$. Then $z \in \phi(x)$, $z' \in \phi(y)$, and $z \leq z'$ because $z \lhd z'$.

Step 5. Finally, let $\phi(e) \neq \{e\}$, and let $\phi' : X \twoheadrightarrow X$ coincide with ϕ on $X \backslash \{e\}$, and take the value $\{e\}$ on e. By step 4 there is an order $\leq$ on X such that e is its largest element and such that ϕ' is weakly increasing. Now we check that ϕ is weakly increasing in this order as well. We only need to prove that if $x \in X$ then there is $z \in \phi(x)$ and $z' \in \phi(e)$ with $z \leq z'$. But, since $e \in \phi(e)$ and e is the largest element in X, we can set $z' = e$ and be done. ■

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