Universidad de la República
Facultad de Ciencias Sociales
DEPARTAMENTO DE ECONOMIA

Documentos de trabajo

Notes about uniqueness of equilibrium for infinite dimensional economies

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Documento No. 10/93
Diciembre, 1993

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Documento No. 10/93 Octubre, 1993

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Resumen

En el presente artículo presentamos condiciones suficientes para la unicidad del equilibrio walrasiano, para economías con espacios de consumo de dimensión infinita.
Mediante la función exceso de utilidad, transformamos el problema de hallar condiciones de unicidad en un espacio de dimensión infinita, en un problema equivalente en un espacio de dimensión finita.
Las propiedades de dicha función permiten usar técnicas conocidas de la topología diferencial, para obtener condiciones de unicidad.
La función exceso de utilidad aparece como una herramienta poderosa para caracterizar el conjunto de los equilibrios, especialmente en aquellos espacios donde la función exceso de demanda no surge como un resultado natural del proceso de maximización.

1 Introduction

Conditions for uniqueness of equilibrium in economies with finite dimensional consumption space are well know, while uniqueness result in economies with infinite dimensional consumption space are scarcer.

Dana (1991) was the first one to obtain an uniqueness result in infinite dimensional consumption spaces. She considers the case of a pure exchange economy where the agent's consumption space is $L^p_+(\mu)$ and agents have additively separable utilities.

We consider a pure exchange economy with consumptions spaces that are a finite cartesian product of measurable functions. Utility functions are additively separable (see Section 1).

The properties of the excess utility function (see Section 2) allow us to apply degree theory to the considered economies. We prove that the cardinality of equilibrium for these economies is odd and we obtain suficient conditions for uniqueness of equilibrium to hold.

By means of the excess utility function, we transforming an infinite dimensional optimization problem in a finite dimensional one.

In section 3 we illustrate the developed theory with some examples.

In the last section the proof of theorems are given.

2 The Model

Let us consider a pure exchange economy with n agents and l at each state of the world.

The state set is a measure space $(\Omega\mathcal{A}, \nu)$.

*I wish to thank Aloisio Araujo, Paulo K. Monteiro and Ricardo Marchesini for useful comments and suggestions and also Rose Anne Dana for sending me her preprint about Uniqueness under Gross Substitute hypothesis.

We assume that each agent k has the same consumption space, $\mathcal{M} = \Pi_{j=1}^{l}\mathcal{M}_j$ where $\mathcal{M}_j$is the space of all nonnegative measurable functions defined on $(\Omega, \mathcal{A}, \nu)$

Following Mas-Colell (1990), we consider the space Λ of the $C^2(R_{++}^l)$ utility functions U. Utilities are strictly monotones, differentiably strictly concaves and propers.

This space is endowed with the topology of C^2 uniform convergence in compact.

More precisely we say that $\{U_n\} \to U$ if for each compact $K \subset R_{++}^l$

$\|U_n - U\|_K =$

$$ess \sup_{s\in\Omega} \max_{z \in K}(|U_n(s,z) - U(s,z)| + |\partial U_n(s,z) - \partial U(s,z)| + |\partial^2 U_n(s,z) - \partial^2 U(s,z)|$$

go to zero with n.

Agent k is characterized by his utility function u_k and by his endowment w_k.

From now on we will work with economies with the following characteristics:

a) The utility functions $u_k : X_k \to R$ are separable and they are represented by

$$u_k(x) = \int_\Omega U_k(s, x(s))d\nu(s) \tag{1}$$

with $x \in \mathcal{M}$,

b) the $U_k(s,.)$ belongs to a fixed compact subset of Λ,

c) the agents' endowments, $w_k \in \mathcal{M}$ are bounded above and bounded away from zero in any component, i.e. there exists ϵ and H with $\epsilon < w_{kj} < H$

The following definitions are standard.

Definition 1 *An allocation of commodities is a list $x = (x_1, \ldots, x_n)$ where $x_j : \Omega \to R^l$; $j = \{1, \ldots, n\}$ is measurable, and $\sum_{i=1}^n x_i(s) \leq \sum_{i=1}^n w_i(s)$*

Definition 2 *A commodity price system is a mesurable function $p : \Omega \to R_{++}^l$, and for any $z \in R^l$ we denote*

$$\langle p, z\rangle = \int_\Omega p(s)z(s)d\nu(s)$$

Definition 3 *The pair (p, x) is an equilibrium if:*

i) x is an allocation,

ii) $\langle p, x_i\rangle \leq \langle p, w_i\rangle < \infty \ \forall\, i$

iii) if $\langle p, z\rangle \leq \langle p, w_i\rangle$ with $z : \Omega \to R_{++}^l$, then

$$\int_\Omega U_i(s, x_i(s))d\nu(s) \geq \int_\Omega U_i(s, z(s))d\nu(s) \ \ \forall\ i.$$

3 The Excess Utility Function

In order to obtain our results we introduce the excess utility function.

For all λ in the $(n-1)$ dimensional open simplex, Δ^{n-1}, and for all s in Ω, there is a solution in R^{nl}_{++}, $x(s,\lambda)$, of the problem

$$\begin{array}{c} \max \sum_i \lambda_i U_i(s, x_i(s)) \\ \text{subject to } \sum_i x_i(s) \leq \sum_i w_i(s) \text{ and } x_i(s) \geq 0. \end{array} \tag{2}$$

See Mas-Colell (1990) for more details.

For this solution, the following identities hold

$$\lambda_i \partial U_i(s, x(s,\lambda), w) = \gamma(s, \lambda, w) \quad \forall\, i = 1, ..., n \text{ and } \forall\, s \in \Omega. \tag{3}$$

Where $\lambda_i \frac{\partial U_i}{\partial x_j} = \gamma_j$ with $i = \{1, \ldots, n\}$ and $j = \{1, \ldots, l\}$

Let us now define the excess utility function.

Definition 4 *We say that* $e : R^n_+ \to R^n$ $e(\lambda) = (e_1(\lambda), ..., e_n(\lambda))$, *with*

$$e_i(\lambda) = \frac{1}{\lambda_i} \int_{\Omega} \gamma(s,\lambda)[x_i(s,\lambda) - w_i(s)]d\nu(s), \; i = 1, \ldots, n. \tag{4}$$

is the excess utility function.

Each equilibrium (p, x) can be characterized by an unique $\lambda \in \Delta^{n-1}$ such that $e(\lambda) = 0$. This is shown in the next proposition

Proposition 1 *The pair* (p, x) *is an equilibrium iff* $p(s) = \gamma(s,\lambda); x(s) = x(s,\lambda)$ *with* $e(\lambda) = 0$.

Proof: For each $\lambda \in \Delta^{n-1}$ there corresponds an unique $\gamma(s,\lambda)$ R^l_{++} and an alocation $x(s) = x(s,\lambda)$ such that (3) follows.

If we call λ an equilibrium whenever $(p(s), x(s))$ is an equilibrium, we have that λ is an equilibrium iff $e(\lambda) = 0$.

Its convenient to introduce the following definition.

Definition 5 *Let us define by* E *the set:*

$$E = \{\lambda \in \Delta^{n-1} : e(\lambda) = 0\}.$$

that will be called, equilibrium set.

As we have that

$$ess \sup_{s \in \Omega} |\partial U_i(s, w_i(s))| < \infty$$

then E is a non empty set, see Araujo-Monteiro (1988)

Let us now consider the product space $\Lambda \times \mathcal{M}$ of the characteristics (U, w), with the C^2 uniform convergence in compact.

That is if $(U_n, w_n) \to (U, w)$ if for each compct $K \in R^l_{++}$

$$\|(U_n, w_n) - (U, w)\|_K \to 0 \text{ with n.}$$

Where $\|(U, w)\|_K = \|(U)\|_K + \|w\| =$

$$ess \sup_{s \in \Omega} \max_K (|U| + |\partial U\| + \|\partial^2 U\| + \|w(s)\|)$$

This is a metrizable space and the induced metric can be taken as:

$$\|(U, w)\|_K = \sum_{N=1}^{\infty} 2^{-N} \frac{\|(U, w)\|_{K_N}}{1 + \|(U, w)\|_{K_N}}.$$

Where

$$K_N = \{z \in R^l : \frac{1}{N} \leq z_i \leq N\}$$

$$\|(U, w)\|_K = \|U\|_{K_N} + \|w\| = ess \sup_{s \in \Omega} \max_{z \in K} (|U| + |\partial U| + |\partial^2 U| + \|w\|)$$

Let Γ be the set of economies with characteristics in $\Lambda \times \mathcal{M}$ such that zero is a regular value of its excess utility function.

Mas-Colell (1990) proves that Γ is open and dense in the set of economies.

From now on we will work with economies in Γ.

As for economies in Γ, zero is a regular value of the $e(\lambda)$ we have for all $\lambda \in E$ that $J(e(\lambda))$ maps $T_\lambda S^{n-1}_{++} \to T_\lambda S^{n-1}_{++}$.

Hence the rank of $J(e(\lambda))$ is $n - 1$.

The determinant of this map is equal to:

$$[\Pi_T J(e(\lambda)] = \begin{bmatrix} J(e(\lambda)) & \lambda \\ \lambda^{tr} & 0 \end{bmatrix}$$

See Mas Collel (1985) B.5.2.

Definition 6 *Then we put* $signJ(e(\lambda)) = (+1) - 1$ *according to whether* $det[\Pi_T J(e(\lambda))](> 0) < 0$

We are now in condition of stating our main result:

Theorem 1 *Consider an economy with infinitely dimensional consumption set, separable utilities satisfying the conditions in section 1), then:*

(1) The cardinality of E is finite and odd,

(2) If $signJ(e(\lambda))$ is constant in E, there exists an unique equilibrium, where $J(e(\lambda))$ denote the Jacobian of the excess utility function.

4 Examples of Economies With Uniqueness

In this section we consider some examples with uniqueness of equilibria.

Let $[J(e(\lambda))]_{ij}$ be the term in the row i and column j of the Jacobian of the excess utility function.

$$[J(e(\lambda))]_{ij} = \frac{\partial e_i(\lambda)}{\partial \lambda_j}.$$

Then

$$[J(e(\lambda))]_{ij} = \int_\Omega \frac{\partial \{\partial U_i(s, x_i(s,\lambda)) [x_i(s,\lambda) - w_i(s)]\}}{\partial \lambda_j} d\nu(s) \tag{5}$$

where $\partial U_i = \left(\frac{\partial U_i}{\partial x_1}, ..., \frac{\partial U_i}{\partial x_l}\right)$ and $x_i(s,\lambda) = (x_{i1}(s,\lambda), ..., x_{il}(s,\lambda))$.

We have that $\frac{\partial\{\partial U_i\}}{\partial \lambda_j} = \left[\frac{\partial x_i(\lambda)}{\partial \lambda_j}\right] \partial^2 U_i$ with $\left[\frac{\partial x_i(\lambda)}{\partial \lambda_j}\right] = \left(\frac{\partial x_{i1}}{\partial \lambda_j}, ..., \frac{\partial x_{il}}{\partial \lambda_j}\right)$ and

$$\partial^2 U_i = \begin{bmatrix} \partial^2 U_i/\partial {x_1}^2 & \partial^2 U_i/\partial x_1 \partial x_2 & \cdots & \partial^2 U_i/\partial x_1 \partial x_l \\ \partial^2 U_i/\partial x_2 \partial x_1 & \partial^2 U_i/\partial {x_2}^2 & \cdots & \partial^2 U_i/\partial x_2 \partial x_l \\ \vdots & \vdots & & \vdots \\ \partial^2 U_i/\partial x_l \partial x_1 & \partial^2 U_i/\partial x_l \partial x_2 & \cdots & \partial^2 U_i/\partial {x_l}^2 \end{bmatrix}.$$

Then, we obtain

$$[J(e(\lambda))]_{ij} = \int_\Omega \frac{\partial x_i}{\partial \lambda_j} \left[\partial^2 U_i(x_i(s,\lambda)) [x_i(s,\lambda) - w_i(s)]^{tr} + (\partial U_i(s, x_i(s.\lambda))^{tr}\right] d\nu(s). \tag{6}$$

4.1 Economies With Gross Substitute Property.

Following Dana (1991) we say that the excess utility function displays the Gross Substitute property if satisfies the next definition.

Definition 7 *The excess utility function displays the so called "Gross Substitute" property, if:*

$$\frac{\partial e_i(\lambda)}{\partial \lambda_j}(>0) < 0 \quad if\ (i=j)\ i \neq j.$$

Proposition 2 *If the excess utility function has the Gross Substitute property, then* $signJ(e(\lambda))$ *is constant.*

Proof: Let A be $(n-1)\times(n-1)$ northwest submatrix of $\{J(e(\lambda)) + [J(e(\lambda))]^{tr}\}$.

From Gross Substitute property we can prove that A has dominant diagonal positive. See Lugon (1991).

Let $v_n = (0,\dots,1)$, as rank $J(e(\lambda))$ is $(n-1)$, then for all vetor z such that $zv_n = 0$ we have that $zJ(e(\lambda))z > 0$.

Now take $v \neq 0$ with $\lambda v = 0$ and let $v_\alpha = v + \alpha\lambda$; if $\alpha = -\frac{vv_n}{\lambda v_n}$, then $v_\alpha v_n = 0$

If $e(\lambda) = 0$ we have that $v_\alpha J(e(\lambda))v_\alpha = vJ(e(\lambda)) > 0$.

That is if $e(\lambda) = 0$ then $J(e(\lambda))$ as a map from T_λ to T_λ its determinant has sign $(-1)^{n-1}$.

Now the Theorem 1 guarantees the uniqueness.

4.1.1 Economies With One Good in Each State.

Economies with one good in each state of the world and utility functions with the next property,

$$\frac{\partial^2 U}{\partial x^2}(x-w) + \frac{\partial U}{\partial x} \geq 0, \qquad (*)$$

have Gross Substitute property. See Dana (1991)

For the following two examples the above condition is satisfied.

Example 1 *Suppose an economy with individual's utilities :*

$$u_i(x) = \int_\Omega U_i(x(s))g_i(s)d\mu(s), \ with\ g_i : \Omega \to R^+ \ and\ i = \{1,\dots,n\}.$$

If $U_i(x)$ *satisfy (*) then we have uniqueness.*

For instance: $u_i(x) = \int_\Omega x(s)_i^\alpha e^{-rs}d\mu(s)$ with $0 < \alpha < 1$ *and* $r > 0$

Example 2 *Let us now consider economies with the following utilities:*

$$u_i(x) = \int_\Omega [a_i(s) + b_i(s)x(s)]^{\alpha_i}d\mu(s).$$

Where $a_i(s) > -w(s)$ $b_i(s) > 0$ *and* $0 < \alpha_i < 1$

For these economies (*) is satiesfied.

Remark: The property $\partial[x\partial U] \geq 0$ is equivalent to risk aversion smaller than one.

Example 3 *If the economy has one good in each state of the world i.e. $x : \Omega \to R$, and If each agent has risk aversion is smaller than one, then the economy display Gross Substitute property.*

Remark: When $x(s,\cdot) : \Delta \to R$, the fact that

$$x_i(s, \lambda_1, ..., \lambda_{k-1}, \cdot, \lambda_{k+1}, ..., \lambda_n)$$

is increasing for λ_i and decreasing if λ_k, $k \neq i$ has economical sense because $x(\lambda)$ is a solution of the social choice problem.

If the social weight of $i-th$ agent is increased, then the consumption bundle of the agent must also increase.

4.2 Separable Goods Economies

Consider economies with good-separable utility functions i.e.

$$\frac{\partial^2 U_h}{\partial x^i \partial x^k} = 0 \quad \forall h \in \{1, ..., n\} \text{ and } i,\ k\ \in \{1, ..., l\},\ i\ \neq\ k.$$

Proposition 3 *Let U be a utility function that is both additively separable and good separable. If*

$$\partial^2 U_h(s, x_h(s,\lambda))[x_h(s,\lambda) - w_h(s)]^{tr} + [\partial U_h(s, x_h(s,\lambda))]^{tr} \gg 0 \ \ (\ll 0) \tag{7}$$

$$\forall h \in \{1, ..., n\} and\, s \in \Omega,$$

then we have uniqueness.

Proof:

From the first order conditions, (2), we have that

$$\lambda_1 \partial U_1(s, x_1(s,\lambda)) \equiv \cdots \equiv \lambda_n \partial U_n(s, x_n(s,\lambda))$$

that is

$$\lambda_1 \frac{\partial U_1}{\partial x^k} \equiv \cdots \equiv \lambda_n \frac{\partial U_n}{\partial x^k} \quad \forall k\ \in \{1, ..., l\}$$

where

$$x_h = (x_h^1, ... x_h^l) \ \ 1 \leq h \leq n.$$

Taking derivatives with respect to λ_j ($j \in \{1, ..., n\}$,) and recalling that $\partial^2 U_h / \partial x^i \partial x^k = 0$ it follows that

$$\lambda_1 a_{1k} \frac{\partial x_1^k}{\partial \lambda_j} = \cdots = \lambda_j a_{jk} \frac{\partial x_j^k}{\partial \lambda_j} + b_{jk} = \cdots = \lambda_n a_{nk} \frac{\partial x_n^k}{\partial \lambda_j} \tag{8}$$

where

$$a_{hk} = \frac{\partial^2 U_h}{\partial x^{k^2}} \text{ and } b_{hk} = \frac{\partial U_h}{\partial x^k}.$$

Let $w^k(s)$ be the total endowment of good k.

From $x_1^k(s,\lambda) + ... + x_n^k(s,\lambda) = w^k(s)$ we obtain that

$$\frac{\partial x_1^k}{\partial \lambda_j} + \cdots + \frac{\partial x_n^k}{\partial \lambda_j} = 0. \tag{9}$$

From (8) we obtain the following equation

$$\frac{\lambda_1 a_{1k}}{\lambda_h a_{hk}} \frac{\partial x_1^k}{\partial \lambda_j} = \frac{\partial x_h^k}{\partial \lambda_j}, \quad \forall h \neq j. \tag{10}$$

Replacing (10) in (9) and without loss of generality supposing that $j \neq 1$ and $j \neq h \neq 1$ give us

$$\frac{\partial x_1^k}{\partial \lambda_j} + \frac{\partial x_j^k}{\partial \lambda_j} + \frac{\partial x_1^k}{\partial \lambda_j} \left\{ \sum_{h \neq 1\ h \neq j} \frac{1}{\lambda_h a_{hk}} \right\} \lambda_1 a_{1k} = 0$$

or equivalently

$$\frac{\partial x_j^k}{\partial \lambda_j} + \frac{\partial x_1^k}{\partial \lambda_j} \left\{ \sum_{h \neq j} \frac{1}{\lambda_h a_{hk}} \right\} \lambda_1 a_{1k} = 0. \tag{11}$$

From (8)

$$-\lambda_1 a_{1k} \frac{\partial x_1^k}{\partial \lambda_j} + \lambda_j a_{jk} \frac{\partial x_j^k}{\partial \lambda_j} = -b_{jk}. \tag{12}$$

Finally, replacing (12) in (11) we obtain

$$\frac{\partial x_1^k}{\partial \lambda_j} = \frac{b_{jk}}{\lambda_j a_{jk} \lambda_1 a_{1k} \sum_h \frac{1}{\lambda_h a_{hk}}} < 0$$

$$\frac{\partial x_i^k}{\partial \lambda_j} < 0 \quad, \forall k \text{ and } i \neq j. \tag{13}$$

From the fact that $x(\lambda)$ is homogeneous of degree zero we obtain that

$$\frac{\partial x_j^k}{\partial \lambda_j} > 0 \ \ \forall k \text{ and } j. \tag{14}$$

Then (13) and (14) are sufficient conditions to obtain (8) of Proposition 2.

The following example ilustrates this proposition.

Example 4 *Economies with utility functions such that*

$$\partial \left[x \frac{\partial U_i(s,x)}{\partial x} \right] \geq 0 \tag{15}$$

have a unique equilibrium price.

In order to prove the proposition recall that:

1) $w_i(s)$ is positive for all i and $s \in \Omega$,

2) the Hessian is a diagonal matrix with negative entries.

For instance, economies with the following utility functions

$$U(x) = \int_\Omega [\sum_{j=0}^{N} \rho^j u_j(x_j)] d\mu(s)$$

with $x = x_1, \ldots, x_n$ $u_j(x_j) = x_j^{\alpha_j}$ and $0 < \rho < 1$,$0 < \alpha_j < 1$
Then we have Gross Substitute property.

4.3 Economies With Two Goods and Two Agents.

Let E be an economy with two goods an two agents, (u_i, w_i), $i = \{1,2\}$

From the first order condition we have that:

$$\lambda_1 \partial U_i(x_1) \equiv \lambda_2 \partial U_2(x_2)$$

where $x_k = (x_k^1, x_k^2)$. Taking the derivative with respect to λ_1 in the above identity we obtain:

$$b_1^i + \lambda_1 a_1^{i1} \frac{\partial x_1^1}{\partial \lambda_1} + \lambda_1 a_1^{i2} \frac{\partial x_1^2}{\partial \lambda_1} = \lambda_2 a_2^{i1} \frac{\partial x_2^1}{\partial \lambda_1} + \lambda_2 a_2^{i2} \frac{\partial x_2^2}{\partial \lambda_1} \quad i = 1,2 \tag{16}$$

where

$$a_k^{ij} = \frac{\partial^2 U_k}{\partial x^i \partial x^j} \quad \text{and} \quad b_1^i = \frac{\partial U_1}{\partial x^i}; \quad i,j,k = 1,2.$$

Let $w^i(s)$ be the endowment of good i. Then

$$x_1^i(s,\lambda) + x_2^i(s,\lambda) = w^i(s). \tag{17}$$

Hence

$$\frac{\partial x_1^i}{\partial \lambda_j} = -\frac{\partial x_2^i}{\partial \lambda_j}. \tag{18}$$

Replacing (18) in (16); the below equations follows

$$(\lambda_1 a_1^{11} + \lambda_2 a_2^{11})\frac{\partial x_1^1}{\partial \lambda_1} + (\lambda_1 a_1^{12} + \lambda_2 a_2^{12})\frac{\partial x_1^2}{\partial \lambda_1} = -b_1^1$$

$$(\lambda_1 a_1^{21} + \lambda_2 a_2^{21})\frac{\partial x_1^1}{\partial \lambda_1} + (\lambda_2 a_1^{22} + \lambda_2 a_2^{22})\frac{\partial x_1^2}{\partial \lambda_1} = -b_1^2.$$

Then

$$\frac{\partial x_1^1}{\partial \lambda_1} = \frac{1}{\nabla}\begin{vmatrix} -b_1^1 & \lambda_1 a_1^{12} + \lambda_2 a_2^{12} \\ -b_1^2 & \lambda_1 a_1^{22} + \lambda_2 a_2^{22} \end{vmatrix} \tag{19}$$

and

$$\frac{\partial x_1^2}{\partial \lambda_1} = \frac{1}{\nabla}\begin{vmatrix} \lambda_1 a_1^{11} + \lambda_2 a_2^{11} & -b_1^1 \\ \lambda_1 a_1^{21} + \lambda_2 a_2^{21} & -b_1^2 \end{vmatrix} \tag{20}$$

where ∇ is the determinant of a 2×2 hessian matrix of a convex combination of differntiably strictly concave functions, then ∇ is non negative. We suppose that $\nabla > 0$.

From the fact that $x(\lambda)$ is homogeneous of degree zero [see Property 2,(Sec 1)] we obtain that

$$sgn\frac{\partial x_j^i}{\partial \lambda_j} = -sgn\frac{\partial x_j^i}{\partial \lambda_k} \quad i = 1,2 \text{ and } j \neq k. \tag{21}$$

From (17), we obtain that:

$$\frac{\partial x_j^i}{\partial \lambda_j} = -\frac{\partial x_k^i}{\partial \lambda_j} \quad i = 1,2 \text{ and } j \neq k. \tag{22}$$

The next proposition follows.

Proposition 4 *If* $\frac{\partial x_j^i}{\partial \lambda_j}$ $\forall\ i = 1,2$ *and* $j = 1,2$ *has the same sign and if*

$$sgn\left[\partial^2 U_i.(x^i - w_i) + \partial U_i\right] \text{ be constant,} \tag{23}$$

then uniqueness follows.

Proof: In this conditions Gross Substitute property follows..

A suficient condition to obtain (23) is that:

$$\left[\partial^2 U_i.w_i\right] \leq 0 \text{ and } \partial\left[x\frac{\partial U_i(s,x)}{\partial x}\right] \geq 0$$

Example 5 *Economics with* $a_k^{ij} > 0 i \neq j$, *and* $k = 1,2$ *satisfying (23), have uniqueness of equilibrium.*

Example 6 *Suppose an economy with*

$$u_1(X) = \int_\Omega (a(s)x(s) + b(s)y(s))^\alpha d\mu(s)$$

$$u_2(X) = \int_\Omega (x(s)^\beta + y(s)^\gamma) d\mu(s).$$

Where $X = (x, y)$ $0 < \{\alpha, \beta, \gamma\} < 1$ *with* a *and* b *integrable functions* : $\Omega \rightarrow R^{++}$.
The endowments are $w_i = \{w_{ix}, w_{iy}\}$
We obtain that

$$\partial^2 U_2.(x) + \partial U_2 = \alpha^2(ax + by)^{\alpha-1}\{a, b\} > 0$$
$$\partial^2 U_2 w^2 = \alpha(\alpha - 1)(ax + by)^{\alpha-2}\{aw_1 + bw_2, bw_1 + aw_2\} \leq 0$$

Then

$$[\partial^2 U_2](x - w) + \partial U_2 > 0$$

From (19) and (20) it follows that

$$\frac{\partial y_1}{\partial \lambda_1} = -\frac{b}{\nabla}\alpha(ax + by)^{\alpha-1}\lambda_2\gamma(\gamma - 1)y_2^{\gamma-2} > 0$$

$$\frac{\partial x_1}{\partial \lambda_1} = -\frac{a}{\nabla}\alpha(ax_1 + by_1)^{\alpha-1}\lambda_2\beta(\beta - 1)x_1^{\beta-2} > 0$$

Those conditions are stisfied for u_2 *because it is a separable utility function. Uniqueness follows.*

Example 7 *The same result is obtained with*

$$u_1(X) = \int_\Omega log[a(s)x(s) + b(s)y(s)d\mu(s).$$

and u_2 *a separable utility function.*

5 Proof of Theorems

In order to prove those theorems, we need the following lemmas:

Lemma 1 *The excess utility function is* C^1.

Proof:

$\gamma(s,\lambda,w)$ and $x_i(s,\lambda,w)$ are C^1 with with respect to λ

To see that this is truth, observe that if $x(s,\lambda)$ is a solution for (2) then we have the following identity:

$$\sum_{i=1}^{n} x_i(s,\lambda,w) = \sum_{i=1}^{n} w_i(s). \tag{24}$$

Now let us consider the sistem of equations:

$$\begin{array}{ll} \lambda_i \partial U_i(s, x(s,\lambda), w) = \gamma(s,\lambda,w) & (3) \\ \sum_{i=1}^{n} x_i(s,\lambda,w) = \sum_{i=1}^{n} w_i(s). & (4) \end{array}$$

From the implicit function theorem, taking derivatives in the above sistem, with respect to x and γ, we obtain a matrix with the following form:

$$M = \begin{bmatrix} A & B \\ B^t & 0 \end{bmatrix}$$

Where A is a$(nl) \times (nl)$ matrix,

$$A = \begin{bmatrix} U_{11}^1 & \cdots & U_{l1}^1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 & \cdots & \cdots & 0 \\ U_{1l}^1 & \cdots & U_{ll}^1 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & \ddots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & U_{11}^n & \cdots & U_{l1}^n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & U_{l1}^n & \cdots & U_{ll}^n \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

That is B is a $l \times (nl)$ matrix

Claim There is not a vector $z = (v, w) \neq 0$ such that $Mz = 0$.

Proof Let v such that $Mz = 0$, then

$$B^{tr} v = 0 \tag{25}$$

and

$$Av + Bw = 0 \tag{26}$$

Then for (4) and (5), we have that

$$v^{tr} Av = 0 \tag{27}$$

If $v \in ker B^{tr}$ then

$$\begin{array}{cccc} v_1 + & v_{l+1} + & \cdots + & v_{(n-1)l+1} = 0 \\ v_2 + & v_{l+2} + & \cdots + & v_{(n-1)l+2} = 0 \\ \vdots & \vdots & \vdots & \vdots \\ v_l + & v_{2l} + & \cdots + & v_{nl} \ = \ 0 \end{array}$$

Observe that

$$\partial \sum_{i=1}^{n} \lambda^i U^i =$$

$$\{\lambda^1 \frac{\partial U^1}{\partial x_1}, \lambda^1 \frac{\partial U^1}{\partial x_2}, \ldots, \lambda^1 \frac{\partial U^1}{\partial x_l}, \cdots, \lambda^n \frac{\partial U^n}{\partial x_1}, \lambda^n \frac{\partial U^n}{\partial x_2}, \ldots, \lambda^n \frac{\partial U^n}{\partial x_l}\} =$$

$$= \{\gamma_1, \gamma_2, \ldots, \gamma_l, \cdots, \gamma_1, \gamma_2, \ldots, \gamma_l, \cdots, \gamma_1, \gamma_2, \ldots, \gamma_l\}$$

Then

$$\partial\{\sum_{i=1}^{n} \lambda^i U^i\}.v = \gamma_1(v_1 + v_{l+1} + \ldots + v_{n-1)l+1}) + \cdots + \gamma_l(v_l + v_{2l} + \ldots + v_{nl}) = 0 \tag{28}$$

Because $\sum_{i=1}^{n} \lambda^i U^i$ is diferentiably strictly convex, (6) and (7) we have that $v = 0$.
B is a injective matrix then, from (6) $w = 0$.
We have that $z = 0$. Proving our claim.
From the claim and the fact that $(U_i(s, \cdot)$, is in an compact set of Λ, the lemma follows.

Lemma 2 *The excess utility function has the following properties:*

1) $e(\lambda)$ *is homogeneous of degree zero;*

2) $\lambda e(\lambda) = 0,\ \forall\, \lambda \in R^n_{++}$*;*

3) there exists $k \in R$ *such that* $e(\lambda) \ll k\mathbf{1}$.

4) $||e(\lambda)|| \to \infty$ *as* $\lambda_i \to 0$ *for any* $i \in \{1, ..., n\}$*;*

5) $\partial e(\lambda) : T_p(s)^{n-1} \to T_p(s)^{n-1}$.

Proof: To prove 1) note that $x_i(s, w, \cdot)$ is homogeneous of degree zero $\forall si \in \Omega$.

Properties 2, and 4 are immediate. Property 5 follows from the fact that zero is a regular value of e.

To prove Property 3, note that from equation (2) we can write

$$e_i(\lambda) = \int_\Omega \partial U_i(s, x_i(\lambda))[x_i(s, \lambda) - w_i(s)]d\nu(s).$$

From the concavity of U_i it follows that:

$$U_i(s, x(s, \lambda)) - U_i(s, w(s)) \geq \partial U_i(s, x(s, \lambda))(x_i(s, \lambda) - w(s)).$$

Therefore,

$$e_i(\lambda) \leq \int_\Omega U_i(s, x_i(s, \lambda)) - U_i(w_i(s))\, d\nu(s) \leq \int_\Omega U_i(\sum_{j=1}^{n} w_j(s))\, d\nu(s),\ \forall\, \lambda.$$

If we let

$$k_i = \int_\Omega U_i(\sum_{i=1}^{n} w_i(s))\, d\nu(s) \text{ and } k = \sup_{1 \leq i \leq n} k_i$$

Property 3) follows.

We can now prove the following lemma:

Lemma 3 : *The excess utility function is an outward pointing vector field on the tangent space of* $S^{n-1}_{++} = \{\lambda \in R^n_{++} : ||\lambda|| = 1\}$.

Proof: With a straightforward application of Property 2 of lemma (2) we obtain that for all $\lambda \in S^{n-1}_{++}$, $e(\lambda) \in T_\lambda S$.

To prove that $e(\lambda)$ is an outward pointing vector field, let us now define z_i

$$z_i = \lim_{\lambda_m \to \lambda \in \partial S^{n-1}_{++}} \frac{e_i(\lambda_m)}{||e(\lambda_m)||}.$$

By Property 3 we know that there exists $k \in R$ such that $e_i(\lambda) \leq k$ and by Property 4, $||e(\lambda)|| \to \infty$. Then we conclude that $z_i \leq 0$.

Furthermore, z_i could be different from zero only if λ_i were zero. This follows from the fact that if λ_i is different from zero, then we can write

$$e_i(\lambda_m) = \frac{-1}{\lambda_{mi}} \sum_{j \neq i} e_j(\lambda_m) \geq -\frac{k^*}{\lambda_{mi}}.$$

Letting $k' = -k^*/\lambda_{mi}$, we have that $k' \leq e_i(\lambda_m) \leq k$. Hence $z_i = 0$.

Strictly speaking, we have proved that we have a continuous outward pointing vector field for *almost any point* in the boundary of S^{n-1}_{++}. The excess utility function has similar properties to those of the excess demand function. Mas-Colell (1985) proves that for excess demand functions there is an homotopic inward vector field for *all points* of the boundary S^{n-1}_{++}. In our case, with an analogous proof, we can obtain an homotopic outward vector field for excess utility functions.

Proof of Theorem 1

From the fact that $e(\lambda)$ is homogeneous of degree zero we can define the equilibrium set as

$$E = \{\lambda \in S^{n-1}_{++} : e(\lambda) = 0\}.$$

From property 4) in lemma (2) and from the continuity of e we have that if $\{\lambda_n\} \in E$, and $\lambda_n \to \lambda$, then $\lambda \in E$.

Then E is a compact set in R^{n-1}.

Moreover, from the fact that zero is a regular value of e, we have that E is a finite set.

Since S^{n-1}_{++} is homeomrphic to the $(n-1)$-dimensional disk, its Euler characteristic is one.

On the other hand, $e(\lambda)$ is an outward vector field on the tangent space of S^{n-1}_{++}, then item 1) follows, from the Poincaré-Hopf theorem.

Now item 2) is a straightforward aplications of the Poincaré-Hopf theorem.

Because:

$$1 = \sum_{\{\lambda : e(\lambda = 0)\}} signdet J(e(\lambda)).$$

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