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A full characterization of representable preferences

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Abstract

We fully characterize the preference relations that can be represented by a utility. Representation is equivalent to the condition that preferences do not have too many "jumps". A characterization of preferences that can be represented by a continuous utility follows trivially from our non-topological characterization.

Resúmen

Se caracterizan las relaciones de preferencias que se pueden representar mediante una función de utilidad. La calidad de representable es equivalente a que la relación de preferencias no tenga demasiados "saltos". Una caracterización de las preferencias que se pueden representar mediante una función contínua sigue consecuencia trivial de nuestra caracterización no-topológica.

JEL Classification: D00, D11, D50, C70.

1 Introduction

A set of players, and a strategy space and preference relation for each of them, describe a normal-form game. Preferences are generally specified by a payoff function instead of a preference relation over strategy profiles. Are these two equivalent? In other words, does the assumption of a payoff function alone—without further structure such as continuity or smoothness—imply something about the players' preferences? In this paper we address this question in a general setting. We find a condition that the preference relation must satisfy in order to be representable by a utility function. Moreover, this condition also guarantees the existence of a utility. That is, we give necessary and sufficient conditions for preference relations to be represented by a utility function, and for preferences to be represented by a continuous utility function. The necessity parts of the theorems are new.[1] The condition, that we call the Cantor-Debreu axiom, states that the preference relation cannot have too many "jumps".

The results are important for three reasons. First, necessary and sufficient conditions for a representation with a not-necessarily-continuous utility function is relevant because, in applications, utility functions are often discontinuous. For example, in many auction models payoffs are discontinuous in bids. There is also an extensive literature on games with discontinuous payoffs. In addition, the issue of whether a preference relation is representable by a utility function is conceptually different from that of whether it is representable by a continuous utility function.

Second, necessary and sufficient conditions for a representation with a not-necessarily-continuous utility function is also relevant from a theoretical point of view: the characterization of preferences representable by a continuous utility function is a trivial consequence of our non-topological characterization. This corollary of our non-topological characterization is relevant because economists are often concerned with maximization problems, where continuity of the objective function is important.

[1] The related literature will be discussed later. It will suffice here to say that the only paper related to our necessity result is that of Beardon et al (2000).

Finally, as was argued earlier, necessary conditions for representations are relevant because one needs to know what is being assumed about a decision maker when his preferences are given by a utility function.[2]

2 The Results

A set X is **ordered** if there is a complete, transitive binary relation $\preceq$ on X. A real-valued function u on an ordered space X is **increasing** if and only if

$$x \preceq y \Leftrightarrow u(x) \leq u(y) \quad \text{for all } x, y \in X$$

An order $\preceq$ on X satisfies the **Cantor-Debreu (CD) Axiom** if and only if there exists a countable set $Z \subseteq X$ such that if x and y in X satisfy $x \prec y$, then there exists a z in Z satisfying $x \preceq z \preceq y$.

This axiom was first used by Debreu (1954), and is closely related to a postulate used in Cantor (as quoted by Debreu, 1954). We now give our characterization of orders that can be represented by a utility function.

Theorem A: *Let X be an ordered set. The order $\preceq$ satisfies the Cantor-Debreu axiom if and only if there exists an increasing function $u : X \to \mathbf{R}$.*

Proposition 5 in Debreu (1964) shows that the CD axiom implies the existence of an increasing extended-real valued function, but it is immediate from his proof that the function must be finite. The converse is new. The closest result to the *only if* part of the Theorem is that of Beardon et al (2000). They show that every non-representable chain satisfies one of four conditions. The conditions are that the chain be long, or planar, or an Aronszajn-like Chain, or a Souslin Chain.

Example 1: The Lexicographic order on $\mathbf{R}^2$. The lexicographic order is defined as follows: $(x_1, x_2) \prec (y_1, y_2)$ iff $x_1 < y_1$ or $x_1 = y_1$ and $x_2 < y_2$. To see *why* this order does

[2]The importance of this point was highlighted to one of us by Larry Epstein in a similar context.

not have a representation, we now show that it fails to satisfy the necessary condition for a representation. For any arbitrary real numbers p, q and r such that $p < q$, $(r, p) \prec (r, q)$, so if $\preceq$ satisfied the CD axiom and Z were the countable set in the axiom, Z should contain an element (r, s) for $p \leq s \leq q$, an absurdity, since r was an arbitrary real. ■

Example 2: Monotone and Continuous preferences in $\mathbf{R}^n$ (Wold, 1943). An order $\preceq$ on $\mathbf{R}^n$ is monotonic if and only if $x \ll y$ implies $x \prec y$. We now show that continuous and monotonic preferences satisfy the CD axiom. For any x, y such that $x \prec y$ there exists $\alpha > 0$ such that, for $e = (1, ..., 1)$, $x + \alpha e \sim y$. We have that $x \prec x + \frac{\alpha}{2}e \prec y$, so there exists a neighborhood of $x + \frac{\alpha}{2}e$ such that for all z in the neighborhood, $x \prec z \prec y$. Since $\mathbf{Q}^n$, the set of vectors with rational components, is dense in $\mathbf{R}^n$, there must be some z of $\mathbf{Q}^n$ in that neighborhood. ■

Theorem B provides a topological characterization of the CD axiom—it translates the CD axiom in terms of separability. This is important because many representation theorems require that X is a separable space, which may give the impression that representation follows from the topological structure of the space involved. Theorem B suggests that separability is a way of using the CD axiom.

Let X be a topological space. An order $\preceq$ on X is **continuous** if and only if, for all x' in X, the sets $\{x : x \preceq x'\}$ and $\{x : x' \preceq x\}$ are closed. On the other hand, if X is ordered by $\preceq$, a topology on X is **natural** if and only if $\preceq$ is continuous. The **order-interval topology** is the coarsest natural topology.

Let $X/\!\sim$ denote the set of indifference classes of X. For $a \in X/\!\sim$, and $x \in X$, $x \preceq a$ will mean $x \preceq y$ for all y in a. The order $\preceq$ has a **countable number of gaps** if and only if the set

$$\{(a, b) \in (X/\!\sim)^2 : a \prec b \text{ and } X = \{x : x \preceq a\} \cup \{x : b \preceq x\}\}$$

is at most countably infinite.

Theorem B: *Let X be an ordered set. The order $\preceq$ satisfies the Cantor-Debreu axiom if and only if X, endowed with the order interval topology, is separable and $\preceq$ has a countable* *number of gaps.* ■

Remark. *Theorem B together with Theorem A generalize Theorem I in Debreu (1954) by weakening his requirement that the space X be connected to requiring that the order has countably many gaps. This allows the representation of preferences when there are goods that come in integer amounts.*

Finally, we give a full characterization of continuously representable preferences.

Theorem C: *Let X be an ordered topological space. The order $\preceq$ is continuous and satisfies the Cantor-Debreu axiom if and only if there exists a continuous and increasing function $u : X \to R$.*

Sufficient conditions for the existence of a continuous utility were first given in Eilenberg (1941), and then in Debreu (1954). The sufficiency part of the theorem was shown in Lemma II in Debreu (1954). The converse is new, and we do not know of any papers that have dealt with necessary conditions for continuous representations.

3 Proofs

We first prove a simple lemma, which is similar to the standard result in real analysis that a monotone function has at most a countable number of discontinuities. The lemma is different, however, because it involves no topological notions.

Lemma: *Let A be an ordered set such that $\{(x, y) \in A^2 : x \neq y \text{ and } x \sim y\} = \emptyset$. If $v : A \to \mathbf{R}$, is increasing, $\preceq$ has a countable number of gaps.*

Proof. It will suffice to show that if $v : A \to \mathbf{R}$ is increasing, the set

$$D \equiv \left\{(a, b) \in A^2 : a \prec b \text{ and } \{z \in \mathbf{R} : v(a) < z < v(b)\} \cap v(A) = \emptyset\right\}$$

is countable. For $m \in \mathbf{N}$, let $I(m) = \{a \in A : v(a) \in (-m, m)\}$. Since $A = \cup_{m \in \mathbf{N}} I(m)$ it will be enough to show that for an arbitrary m, the set $D^m \equiv D \cap I(m)^2$ is countable.

Define

$$D_n^m \equiv \left\{(a,b) \in D^m : \frac{1}{n} \leq v(b) - v(a)\right\}.$$

and note that $D^m = \cup_{n\in\mathbf{N}} D_n^m$. Let $(a_1,a_2), (a_3,a_4), ..., (a_{k-1},a_k)$ be different elements of D_n^m, for some m, n. The definition of D implies that, for any two different $(a,b), (a',b') \in D_n^m$, either $b \preceq a'$ or $b' \preceq a$. Hence, without loss of generality, let $a_2 \preceq a_3, a_4 \preceq a_5 \ldots a_{k-2} \preceq a_{k-1}$. Note that k is even, so we can write

$$\frac{k}{n} \leq \sum_{i=0}^{(k-2)/2} v(a_{2i+2}) - v(a_{2i+1}) = v(a_k) - v(a_1) + \sum_{i=1}^{(k-2)/2} v(a_{2i}) - v(a_{2i+1}).$$

But, for all $i = 1, \ldots (k-2)/2$, $v(a_{2i}) - v(a_{2i+1}) \leq 0$, so $k/n \leq v(a_k) - v(a_1) \leq 2m$. Hence D_m^n is finite. ■

Proof of Theorem A. Proposition 5 in Debreu (1964) states that if $\preceq$ satisfies the CD axiom, there exists an increasing function $v : X \to \mathbf{R} \cup \{\pm\infty\}$, but its proof shows that v is always real valued, so existence is proved.

Denote by $a(x)$ the indifference class of x in X and let $x : X/\!\sim \to X$ be a function that picks, for each a in $X/\!\sim$, an arbitrary x in a (it exists by the axiom of choice). Define $v : X/\!\sim \to \mathbf{R}$ by $v(a) = u(x(a))$ for all a in $X/\!\sim$. By the lemma, the set

$$D \equiv \left\{(a,b) \in (X/\!\sim)^2 : a \prec b \text{ and } \{z : v(a) < z < v(b)\} \cap v(X/\!\sim) = \emptyset\right\}$$

is countable, so let (a_i, b_i) be an enumeration of D, and define the countable set

$$Z_1 \equiv \{x(a_i) : (a_i, b_i) \in D\}.$$

Since $\mathbf{R}$ is second countable, so is $u(X) \cap \mathbf{R}$, and therefore, $u(X) \cap \mathbf{R}$ is separable. Let S be the countable dense subset in $u(X) \cap \mathbf{R}$. Since v is injective and S is countable, so is $v^{-1}(S) \subseteq X/\!\sim$. Let $Z_2 \equiv \{x(a_i) : a_i \in v^{-1}(S)\}$ and note that Z_2 is countable.

Let $Z = Z_1 \cup Z_2$. For all x, y in X such that $x \prec y$, define the following $u(X)$-relatively open set

$$I \equiv \{z : u(x) < z < u(y)\} \cap u(X)$$

If $I = \emptyset$, there is an $x_i \in Z_1$ such that $x_i \sim x$, so $x \sim x_i \prec y$. If I is nonempty, separability of $u(X)$ implies that there is an $x_i \in Z_2$ such that $x \prec x_i \prec y$. ■

Proof of Theorem B. We first show that the CD axiom implies separability. Let $(x, y) = \{w \in X : x \prec w \prec y\}$, $(x, +\infty) = \{w \in X : x \prec w\}$ and $(-\infty, x) = \{w \in X : w \prec x\}$, for $x, y \in X$. By Theorem A there is an increasing real-valued function u. Let v and the x function be as in the proof of Theorem A. Recall that $u(X)$ is separable and let S be a countable dense subset of $u(X)$. Let $Z_1 = x\left(v^{-1}(S)\right) \subset X$.

If $X/\sim$ has a smallest element denote it by $\inf X/\sim$, and let $x_{\inf} = x\left(\inf X/\sim\right)$. If $X/\sim$ does not have a smallest element let $\{x_{\inf}\} = \emptyset$. Similarly for $x_{\sup}$. Let $Z = Z_1 \cup \{x_{\inf}\} \cup \{x_{\sup}\}$. We shall show that Z is dense by proving that it intersects all the non-empty sets of the form (x, y), $(x, +\infty)$ $(-\infty, x)$ for $x, y \in X$. These sets form a basis for the order-interval topology.

Let $x, y \in X$ such that $(x, y) \neq \emptyset$. Then, $\{r \in \mathbf{R} : u(x) < r < u(y)\} \cap u(X) \neq \emptyset$ is a $u(X)$-relatively open set. Thus there is $z \in Z_1$ such that $z \in (x, y)$.

Let $(x, +\infty) \neq \emptyset$. If $X/\sim$ has a largest element, $x_{\sup} \in (x, +\infty) \cap Z$. If $X/\sim$ does not have a largest element take any $y \in (x, +\infty)$ with $(x, y) \neq \emptyset$. By the argument above, there is $z \in Z_1$ such that $z \in (x, y)$. Similarly when $(-\infty, x) \neq \emptyset$, so the proof of separability is complete.

Note that by Theorem A, there exists an increasing function $v : X/\sim \to \mathbf{R}$. Then, by the lemma, $\preceq$ has countably many gaps.

We now show that if X is separable and $\preceq$ has countably many gaps, it satisfies the CD axiom. Let Z be the countable dense subset in X. Since $\preceq$ has countably many gaps

$$Y \equiv \{x\,(a) : \exists b \text{ such that } a \prec b \text{ and } X = \{x : x \preceq x\,(a)\} \cup \{x : x\,(b) \preceq x\}\}.$$

is countable, and so is $W \equiv Z \cup Y$. Consider a pair x', $y' \in X$ such that $x' \prec y'$. If $\{x : x \preceq x'\} \cup \{x : y' \preceq x\} \neq X$, denseness of Z guarantees that the open interval (x', y') contains an element $z \in Z$. If $\{x : x \preceq x'\} \cup \{x : y' \preceq x\} = X$, $x\,(a\,(x')) \in Y$. Thus, for all $x', y' \in X$ such that $x' \prec y'$, there exists $w \in W$ such that $x' \preceq w \preceq y'$. ■

Proof of Theorem C. The existence of u was shown in Lemma II in Debreu (1954). The converse is a consequence of Theorem A and continuity of u. ■

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