On the dynamics of the Heckscher-Ohlin theory.

Funes

September 30, 2010

1 Introduction

The factor proportions theory is one of the most influential theories of international trade. The special case in which the factors are capital and labor is known as the standard Heckscher-Ohlin theory and is the core of modern international trade theory. This theory is motivated from the observation that countries produce relatively more of the goods which use more intensively the factors in which the countries are more abundant. As many have recognized, any realistic analysis of international trade and growth must take into account that some factors of production—for instance, capital—are produced goods. If we allow countries to optimally accumulate factors, what are the dynamic effects on their comparative advantage, on factor prices, and on the pattern of specialization and growth? To answer these questions one needs to depart from the traditional static analysis and solve the dynamics of trade and growth. This paper builds on early results in the literature and combines the two most influential models of trade and growth, the Neoclassical growth model and the Heckscher-Ohlin model, to understand how trade affects growth.

This paper contributes to the literature by providing closed-form solutions to a two-large-country model and showing that it is tractable and stable[1]. What is crucial in our results is that world prices are determined endogenously in general equilibrium and not taken as given, as in small-country models. I find that while a small country can grow without the retarding force of a terms-of-trade deterioration, a large, capital-intensive rich country can experience terms-of-trade deteriorations, immiserizing growth[2], as a consequence of trading with a large, labor-intensive poor partner. This will depend only on how different factor endowments are across the countries when they start trading[3]. I also show that the model can help to explain why countries experience non-monotonic changes in their pattern of specialization as they grow. A country that at early stages of development was diversified could switch back and forth, becoming specialized and then diversified again, as a consequence of dynamic changes in

[1]Most of the literature has focused on the small-country assumption to answer these questions. In such models the actions of the small country do not affect world prices. When countries are large, in the sense that they affect world prices, we can observe them growing with negative terms of trade.

[2]Immiserizing growth is referred to the situation in which if economic growth is through exports it can lead to a deterioration in the terms of trade (relative price of a country's export to its import) of the exporting country.

[3]Bhagwati (1958) and Johnson (1995) were among the first to realize this possibility in the context of a static Heckscher-Ohlin model. They exogenously considered changes in endowments. We show that this could also be an endogenous outcome of the model.

comparative advantages. On the other hand, if trading partners' factor endowments are close to each other, then we can also observe monotonic changes in the pattern of specialization[4].

The model can also help to explain why large, capital-intensive rich countries can optimally experience periods of booms in consumption. For instance, if a large capital-intensive country suffers a productivity slowdown and starts decumulating capital, and at the same time a large labor-intensive country starts supplying to the world cheaper labor-intensive goods, then the capital-intensive country will optimally decide to consume more. This increase in the demand for labor-intensive goods makes the poor country suffer a terms-of-trade deterioration and overaccumulate capital. The capital stock of the poor country can even overshoot its long-run steady state .

Regarding factor prices, the dynamic model in this paper predicts that, conditional on the initial distribution of factor endowments, there are several possible outcomes: if countries start trading and factor prices are not equalized, factor prices could be equalized in finite time, or we could observe only a tendency towards equalization as the countries move to a specialized steady state. I also show that even in cases where factor endowments are similar, countries might leave the factor price equalization (FPE) set in finite time.

One of the stylized growth facts observed in the data is the lack of convergence across countries. Consistent with the data, the model predicts that if one country has a larger stock of capital at the time it starts trading, it will remain larger along the transition path and at the steady state, and therefore there will be lack of convergence. Also, the model delivers conditional convergence; countries that start further away from their steady state will grow faster, a characteristic observed in newly developed countries in the last decades . More novel, during the transition to the long run, the large, capital-intensive rich country could experience periods of stagnation, while at the same time the large, labor-intensive poor country could enjoy periods of high growth rates. This, and the previous findings, have new policy implications.

For many years economists have been trying to incorporate the implied endogenous dynamics into the standard trade model. Oniki and Uzawa (1965) and Bardhan (1965a, 1965b, 1966) were the first to address this issue where they relied on the simplifying assumption of

[4] The study by Imbs and Wacziarg (2003) presents evidence of non-monotonic changes in the pattern of specialization as countries grow.

the constant savings rate. Later on, Stiglitz (1970) and Deardorff (1973) showed that countries might converge to a lower steady state compared to the autarky steady state if they traded with countries which have different savings rates. Further contributions are Smith (1976, 1977), Findlay (1984), Eaton (1987) and Baldwin (1992).

More recently, Backus, Kehoe and Kydland (1994), Chen (1992), Stokey (1996), Ventura (1997), Jensen and Wang (1997), Mountford (1997), Acemoglu and Ventura (2000), Atkenson and Kehoe (2000), and Gaitan and Roe (2007) have combined versions of the standard Heckscher-Ohlin model with the standard Neoclassical growth model or an overlapping generations model. However, either by assuming that countries remained always diversified (specialized), or by assuming that the structure of production is such that FPE is always the equilibrium outcome, they have not allowed the pattern of specialization to change over time.

Using the static model, researchers have also considered it important to study economies trading under complete specialization. For instance, Leamer (1987) focused on specialized economies according to initial factor endowments. A recent paper by Oslington and Towers (2009) characterizes the set of initial endowments and presents conditions to determine the pattern of specialization between countries in the static model. They apply the model, allowing factor prices not to be equal, to study the impact on wage inequality and factor prices from factor flows.

A seminal paper in this literature is Ventura (1997). He applied the dynamic Heckscher-Ohlin model with linear technologies, therefore factor prices always equalized, to explain several stylized growth facts—for instance, lack of convergence and rapid relative growth of developing countries. By allowing factor prices not to be equalized, which seems to be an empirically relevant case, and with Cobb-Douglas technology, we can also focus on applications over changes in the pattern of specialization as countries grow.

The first paper to address the importance of focusing on dynamics outside the cone of diversification is Cuñat and Maffezzoli (2004). The authors numerically characterize the dynamics of a model with more goods than factors. They focus on dynamics where countries start outside the cone of diversification and present numerical examples for different cases. Because of their assumptions over technologies, they do not find non-monotonic changes in the pattern of specialization as we do. More recently Bajona and Kehoe (2006a, 2006b) showed that given certain conditions of the elasticity of substitution between traded goods, countries can leave the cone of diversification in finite time. We also show that this can happen in a model with Cobb-Douglas technology (unit elasticity) if we allow capital to depreciate over time.

There is substantial evidence of non-FPE across trading partners. Davis and Weinstein (2001, 2003) and Schott (2003) find that the standard Heckscher-Ohlin model's predictions are more in agreement with the data when non-FPE is allowed. Debare and Demiroglu (2003) and Cuñat (2000), based on Deardorff's (1994) lens condition, find evidence of non-FPE between developing and developed countries and FPE for a group of developed countries. Therefore, looking at the implications of the model without FPE might be more empirically relevant than assuming FPE.

The paper is organized as follows. A detailed description of the economy is presented in the next section. After that, the set of initial conditions is divided into four regions, and each region is addressed in a separate subsection. In each of them, it is shown that there is only one equilibrium path heading to an FPE steady state regardless of the timing (region) in which countries start trading. The steady states are characterized and the stability of the system is evaluated for each case. Subsequently, the conditions needed in order to have FPE during the transition are characterized, and the phase diagram of the model is presented. Finally, conclusions are discussed .

2 Model

Consider a standard 2x2x2 Heckscher-Ohlin model. The two countries trading are north and south $i = \{N, S\}$, the two factors of production are capital and labor $K^i(t)$ and L^i, and the two freely tradable goods are intermediate inputs. One of these goods is capital intensive and the other is labor intensive, and they are both combined for the production of a final good in each country. The final good is non-storable and non-tradable, and can be used either for consumption or for investment. The factors of production are mobile across sectors but not between countries. The only goods that are freely tradable internationally are the intermediate goods. Because of this last assumption, the prices of these goods are going to be equated across countries. There is an atomistic household-firm in each country that maximizes the present discounted value of its utility from consuming final goods, $u\left(C^i(t)\right)$

$= \log\left(C^i(t)\right)$. Entering the world with an initial endowment $\left(K^i(0), L^i\right)$, the household-firm decides how much to invest $I^i(t)$, how much to consume $C^i(t)$ and how to allocate capital and labor efficiently across sectors in order to maximize the output of the final good, $Y^i(t)$.

Agents in both countries use the same Cobb-Douglas technology for the production of the final good $Y^i(t) = \left(X_1^i\right)^{\gamma}\left(X_2^i\right)^{1-\gamma}$ where $X_j^i(t)$ are the demands for intermediate inputs $j = \{1, 2\}$. The price of the final good, which the agent takes as given, is normalized to 1. In order to produce intermediate inputs the agent has to combine capital and labor. The technologies for producing these goods are homogeneous of degree one, in particular $Y_j^i(t) = \left(K_j^i\right)^{\theta_j}\left(L_j^i\right)^{1-\theta_j}$. We will assume that agents across different countries use the same technology for the production of a given intermediate input, but that they use different technologies for the production of each intermediate input. Specifically, we will assume that the production of intermediate good 1 is more capital intensive than the production of good 2 $(\theta_1 > \theta_2)$. We also assume no factor intensity reversals, hence, the capital labor ratio employed in the production of the intermediate good 1 is larger than the capital labor ratio employed in the production of intermediate good 2 $\left(K_1^i/L_1^i > K_2^i/L_2^i\right)$ for any relative factor prices. We let $p_j(t)$ be the prices of the intermediate inputs which are taken as given for the agents.

2.1 Household-firm Problem

The representative agent in each country takes prices as given and maximizes the present discounted value of its utility from consuming the unique consumption good c^i produced locally. The prices of the intermediate goods are determined in the world commodity market. If factor supplies are such that FPE holds, we can use the approach suggested by Dixit and Norman (1980) and solve the world planning problem (Integrated World Economy). This is the standard approach taken in the literature to solve this type of problem. For instance, see Bajona and Kehoe (2006) for a description and application of the method. In order to construct the equilibrium, I will focus instead on solving the decentralized problem. This has the advantage that outside the cone, where FPE does not hold and the integrated world equilibrium cannot be solved, it is a tractable problem to solve. Note that inside the cone, it is basically the integrated world equilibrium. The problem of the agent in each country, in intensive form, is the following[5]:

$$\max \int_0^\infty \exp(-\rho t) \log\left(c^i(t)\right) dt$$

subject to:

$$c^i(t) + i^i(t) = G\left(k^i; p\right)$$
$$\dot{k}^i(t) = i^i(t) - \delta k^i(t)$$
$$k^i(0) \text{ given}$$

where δ is the depreciation rate and all the variables are in intensive form, for instance $c^i(t) = \frac{C^i(t)}{L^i}$.

$G\left(k^i; p\right)$ is the solution to the following static sub problem:

$$G\left(k^i; p\right) = \max \left(x_1^i\right)^\gamma \left(x_2^i\right)^{1-\gamma} \tag{1}$$

subject to:

$$\begin{aligned} \sum_j p_j x_j^i &= \sum_j p_j y_j^i && (2) \\ y_j^i &= \left(k_j^i\right)^{\theta_j} \left(l_j^i\right)^{1-\theta_j} \qquad j = \{1, 2\} && (3) \\ k^i &= \sum_j k_j^i && (4) \\ 1 &= \sum_j l_j^i && (5) \\ k_j^i &\geq 0, \ l_j^i \geq 0 && (6) \end{aligned}$$

Note that we are allowing for corner solutions in which one or both of the countries will specialize.

The prices of the intermediate goods $p = \{p_1, p_2\}$ are determined in the world commodity

[5] The problem we are solving is the same as (35) in Bajona and Kehoe (2006a) with Cobb- Douglas technologies and continuous time .

market and are such that the world commodity market clears:

$$\sum_i L^i x^i_j \left(k^i; p\right) = \sum_i L^i y^i_j \left(k^i; p\right) \qquad j = \{1, 2\} \tag{7}$$

where $x^i_j \left(k^i; p\right)$ and $y^i_j \left(k^i; p\right)$ are the optimal conditional demand and supply of intermediate inputs in each country. $G(\cdot)$ is increasing, concave, continuously differentiable and homogeneous of degree one.

Note that while solving the dynamic model at each moment in time we are solving a static sub-problem. This problem can be solved for all possible combinations of factor supplies. By doing so, you can characterize the set of factor supplies such that FPE holds. Also, it will determine the pattern of specialization given factor supplies and prove that prices that decentralize the equilibrium are unique. Moreover, you can also characterize how the equilibrium will look for situations within the cone of diversification, the set of factor supplies such that FPE holds. This is essentially a standard 2x2x2 static Heckscher-Ohlin Model.

The FPE set is:

$$V\left(k^N, k^S\right) = \left\{\left(k^N, k^S\right) \text{ s.t. } \tilde{k}_2 \leq k^N \leq \tilde{k}_1 \text{ and } \tilde{k}_2 \leq k^S \leq \tilde{k}_1\right\} \tag{8}$$

where $k^i = K^i/L^i$, $\tilde{k}_2$ and $\tilde{k}_1$ are capital labor ratios that characterize the lower and upper bound of the FPE set.

The solution for $\tilde{k}_1$ is:

$$\Rightarrow \tilde{k}_1 = \left(\frac{\theta_1}{1-\theta_1}\right)\left(\frac{1-\tilde{\gamma}}{\tilde{\gamma}}\right)\frac{k}{2} \tag{9a}$$

in the same way, for $\tilde{k}_2$:

$$\Rightarrow \tilde{k}_2 = \left(\frac{\theta_2}{1-\theta_2}\right)\left(\frac{1-\tilde{\gamma}}{\tilde{\gamma}}\right)\frac{k}{2} \tag{10}$$

where $\tilde{\gamma} \equiv \gamma\theta_1 + (1-\gamma)\theta_2$, and $k = \sum_i k^i$. These are the same conditions as equations (71), (72) and (73) in Bajona and Kehoe (2006a). Note that both capital labor ratios have to be inside the FPE set in order for there to be FPE. [6] This is the set of relative factor supplies such that FPE holds. The set of factor supplies in which countries are diversifying their

[6] There is an important distinction between countries diversifying and factor prices been equalized. If both countries diversify, then the factor endowments have to belong to the FPE set. However, even if factors do no belong to the set, we could observe one country diversifying and factor prices not been equal. Bajona and Kehoe (2006) make this point very clear with the use of the Lerner Diagram.

production of tradeable goods. Given the assumptions on the technologies in each country, the boundaries of the cone are the same in both countries. Moreover, it is easy to see that if the countries capital labor ratio is inside the cone, these allocations can be attained, hence these are also the optimal capital labor ratios used in the production of the intermediate goods in each country. The larger the difference in factor intensities in the production of both goods, the larger is the set of aggregate capital labor ratios that are consistent with FPE. In the extreme case in which the production of each intermediate input uses only one factor (either $\theta_1 = 1$ and $\theta_2 = 0$, or $\theta_2 = 1$ and $\theta_1 = 0$) the cone is the entire nonnegative orthant. This is the case studied by Ventura (1997).

2.2 Dynamic Model

Let $\tilde{H}^i \equiv H^i\left(c^i(t), k^i(t), q^i(t)\right)$ be the current value Hamiltonian in each country, then:

$$\tilde{H}^i = \log\left(c^i(t)\right) + q^i(t)\left[G\left(k^i(t); p(t)\right) - c^i(t) - \delta k^i(t)\right] \tag{11}$$

where $q^i(t)$ is the current value co-state variable. I am suppressing the dependence of the variables with respect to time in order to have compact notation. The necessary first order conditions are:

$$\begin{aligned}
\left(c^i\right)^{-1} &= q^i && (12)\\
\dot{q}^i &= -\left(G_{k^i}\left(k^i; p\right) - (\rho+\delta)\right) q^i && (13)\\
\dot{k}^i &= G\left(k^i; p\right) - c^i - \delta k^i && (14)\\
\lim_{t\to\infty} e^{-\rho t} k^i q^i &= 0 && (15)
\end{aligned}$$

Where $G^i_{k^i}$ is the partial derivative of total production with respect to k^i, the marginal product of capital in country i.

The following differential equations and the commodity market equilibrium condition (7) characterize the equilibrium dynamics of this model:

$$\begin{aligned}
\dot{c}^N &= c^N \left(G_{k^N} \left(k^N; p \right) - (\rho + \delta) \right) \\
\dot{c}^S &= c^S \left(G_{k^S} \left(k^S; p \right) - (\rho + \delta) \right) \\
\dot{k}^N &= G \left(k^N; p \right) - c^N - \delta k^N \\
\dot{k}^S &= G \left(k^S; p \right) - c^S - \delta k^S
\end{aligned}$$

The challenge is to solve these system of four differential equations in which countries could move in and out the FPE set. Next we will present a description of possible outcomes regarding the pattern of specialization.

2.3 Patterns of Specialization

Starting from an initial distribution of endowments there are four possible patterns of specialization in this model. We refer to them as cases; CASE 1 -FPE holds at $t = 0$, both north and south diversify. CASE 2 - FPE does not hold at $t = 0$, north diversifies and south specializes. CASE 3 - FPE does not hold at $t = 0$, north specializes and south diversifies, CASE 4 - FPE does not hold at $t = 0$, both countries specialize. From the solution to the static model, Grossman Helpman (1991) and Oslington and Towers (2009) show that we can describe in a box all possible patterns of specializations. We replicate these figures here for a world with total capital equal to 10 and total labor equal to 2.

Figure 1 shows the regions in which each case is possible. The goal is to consider every possible distribution of factor supplies. To do this, consider fixing the labor supplies (to make the argument simple, fix labor equal to 1 in both countries), then changing the world capital will only expand the box in one direction (up in the figures above). As we will see in a moment, countries do not converge in factor endowments. Therefore a line cutting the box in half is the set of possible capital labor ratios between the countries (the darker line in the figures is that line were the capital labor ratio in north is always larger than in south). According to the parameter values, this line could intersect different regions of specialization.

In particular, there are two possible outcomes and we will refer to them as outcome I, and outcome II[7]. Outcome I, where two possible patterns of specialization may arise, CASE 1 and CASE 2 (this is the left figure in figure 1); and outcome II where all possible patterns of specialization are present, CASE's 1, 2,3, 4 (this is the right figure in figure 1). The reason is that given assumptions over technologies and initial conditions on labor this imposes a restriction over the possible patterns of specialization, over how different capital labor ratios can differ between countries. Therefore, it is possible to construct an example in which, given assumptions over the technologies an labor, CASES's 3 and 4 may never occur [8].

Figure 2 presents the state space in which only outcome I is feasible, I denote this by State Space I. I divide the $\left(k^N - k^S\right)$ plane into different regions. Regions A and B correspond to the set of factor supplies that belong to the FPE set ($k^i(0) \in V\left(k^N, k^S\right)$). The line dividing regions A and B from C and D is the lower bound of the set. The line dividing region A from B is the set of steady states of the model. In this way, region A considers initial conditions below the steady state while region B considers initial conditions above the steady state. In a moment it is going to be clear why we have a ray of steady states in this model. Regions C and D correspond to the set of factor supplies that do not belong to FPE set ($k^i(0) \notin V\left(k^N, k^S\right)$), and as a consequence FPE does not hold. In a moment it is going to be clear why is it that I divide these regions by a ray and show that the ray is not arbitrary. I will focus on the case in which $k^N(0) > k^S(0)$ which, as we will see in a moment, implies that $k^N(t) > k^S(t)$ for all t. Hence, we can focus only on the lower half space. Since both countries are otherwise identical except for the initial conditions, the dynamics for initial conditions in which $k^N(0) < k^S(0)$ are symmetric to the cases in which $k^N(0) > k^S(0)$. Once we are able to understand what happens when $k^N(0) > k^S(0)$, it will be simply a matter of relabeling the countries to characterize the $k^N(0) < k^S(0)$ cases.

[7] I am abusing of the terminology here. There are two possible outcomes according to parameter values. There is only one state space. State Space I represents one possible outcome and State Space II represents the other possible outcome.

[8] Note that another way to show that there are two cases will be to fix the parameter values and change the distribution of labor between the countries (move the dark line from left to right along the diagonal line). In this case, according to the distribution of labor we can only have the same scenarios as before.

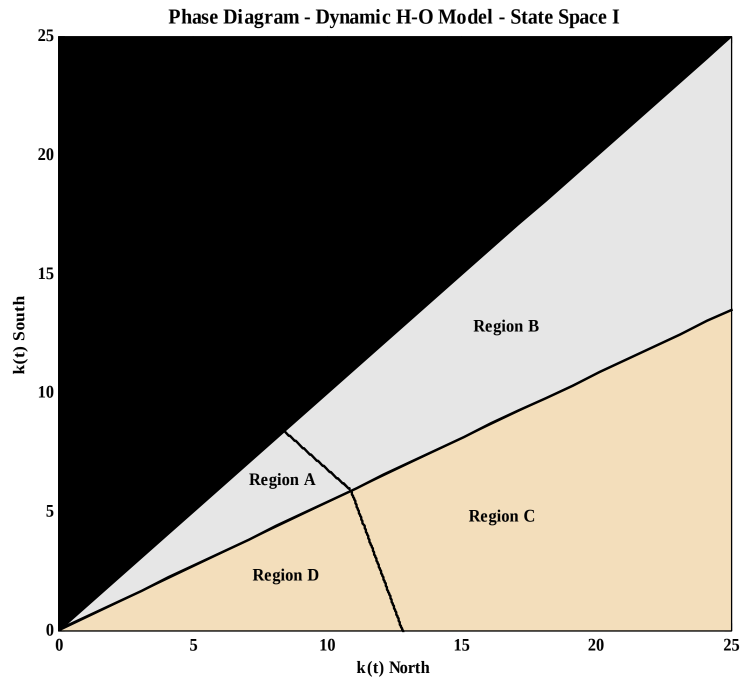

Figure 2

Figure 3 presents the state space where we have outcome II, I denote this by State Space II. Again, I divided the $\left(k^N - k^S\right)$ plane into different regions according to the pattern of specialization. Regions A' and B' correspond to the set of factor supplies that belong to the FPE set $\left(k^i(0) \in V\left(k^N, k^S\right)\right)$. The line dividing regions A' and B' from E is the lower bound of the set. The line dividing region A' from B' is the set of steady states of the model. Region E corresponds to CASE 3, where factor supplies are such that North specializes in the production of the capital intensive good and South diversifies. The ray dividing region E from region G corresponds to the lens condition that determines if there is full specialization or not. Region E corresponds to factor supplies such that both countries specialize, the full specialization case, where North specializes in the production of the capital intensive good and South on the production of the labor intensive good. Finally, the ray dividing region F from region C'D' is the lens that separates CASE 4 from CASE 2. The region C'D' is the region where South is specialized in the production of the labor intensive good and North is diversified. Note that this is CASE 2 and is a region with the same characteristics as regions C and D. in the State Space I

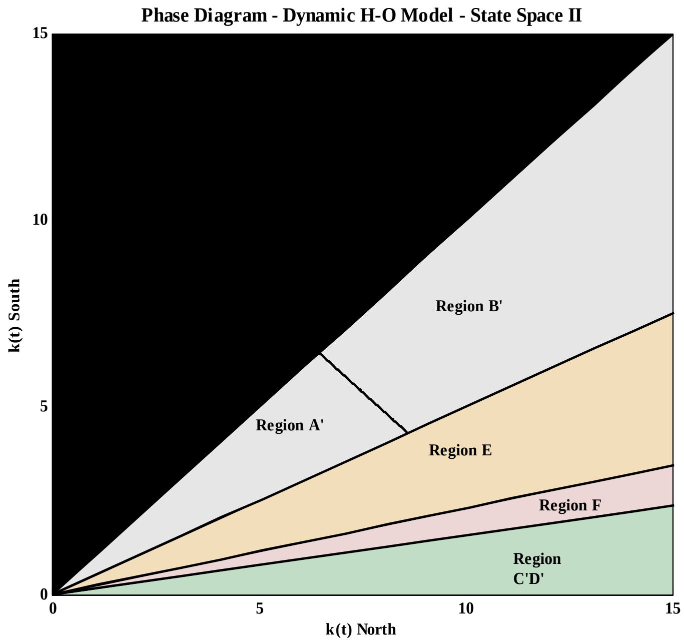

Figure 2

3 Inside the FPE set

Lets assume for the moment that when countries start trading their relative factor supplies belong to the FPE set (Region A (A') and B (B')).

From the necessary first order conditions, and assuming that FPE holds along the transition path, we know that the growth rate of consumption is the same in both countries[9]. This implies that the relative consumption of the representative households in each country are going to stay constant over time. In particular, from the intertemporal budget constraints of

[9] Off course, we will have to verify later if this is an equilibrium. It could be that during the transition one of the countries specializes. We will show in a moment which trajectories are the ones that do.

the agents, we can solve for the initial relative consumption levels:

$$\frac{c^S(0)}{c^N(0)} = \varpi \tag{16}$$

This is a strong result because it implies that, if FPE holds, we do not observe divergence or convergence in consumption levels across countries[10].

The dynamic system governing the economies are a set of four equations, two for each country. Note that by using (16) we can reduce the system to a set of three differential equations, given by:

$$\dot{c}^N = c^N\left(G_{k^N}\left(k^N;p\right) - (\rho+\delta)\right) \tag{17}$$

$$\dot{k}^N = G\left(k^N;p\right) - \delta k^N - c^N \tag{18}$$

$$\dot{k}^S = G\left(k^S;p\right) - \delta k^S - \varpi c^N \tag{19}$$

Since factor prices are equalized $G_{K^N} = G_{K^S}$ and $G_{L^N} = G_{L^S}$. In particular the $G\left(k^i;p\right)$ function is identical in both countries:

$$G\left(k^i;p\right) = \Theta\left(\tilde{\gamma}\frac{k^i}{k/2} + (1-\tilde{\gamma})\right)(k/2)^{\tilde{\gamma}}$$

where Θ is a constant.

3.1 Steady state

Definition: *A steady-state equilibrium is an equilibrium path in which* $k^N(t) = k^{N*}$ *and* $k^S(t) = k^{S*}$. *for all* t.

The steady state of the system has factor prices equalized and there is an infinite number of such steady states. All the variables with "*" are at the steady state. Prices are given by:

$$r^* = \rho + \delta$$

[10] To see this, using the Euler equations in both countries we have that:

$$\frac{c^S(t)}{c^N(t)} = \frac{c^S(0)\, e^{(R(t)-(\rho+\delta))t}}{c^N(0)\, e^{(R(t)-(\rho+\delta))t}} = \varpi$$

where $R(t) = \frac{1}{t}\int_0^t G_k(k(s);p)\,ds$ has the interpretation of an average return on capital in an arbitrary period t..

$$p_1^* = \tilde{\Gamma}(\gamma)^{\frac{1-\theta_1}{1-\tilde{\gamma}}} \left(\frac{r^{*\theta_1} \tilde{\Gamma}(\theta_2)^{1-\theta_1}}{r^{*\theta_2} \tilde{\Gamma}(\theta_1)^{1-\theta_2}} \right)^{\frac{1-\gamma}{1-\tilde{\gamma}}} \tag{20}$$

$$p_2^* = \tilde{\Gamma}(\gamma)^{\frac{1-\theta_2}{1-\tilde{\gamma}}} \left(\frac{r^{*\theta_2} \tilde{\Gamma}(\theta_1)^{1-\theta_2}}{r^{*\theta_1} \tilde{\Gamma}(\theta_2)^{1-\theta_1}} \right)^{\frac{\gamma}{1-\tilde{\gamma}}} \tag{21}$$

$$w^* = (\rho + \delta) \left(\frac{\Psi}{\rho + \delta} \right)^{\frac{1}{1-\tilde{\gamma}}} \tag{22}$$

where r is the rental price of capital, w the wage rate, $\Psi = \tilde{\Gamma}(\gamma) \tilde{\Gamma}(\theta_1)^{\gamma} \tilde{\Gamma}(\theta_2)^{1-\gamma}$ and $\tilde{\Gamma}(x) = x^x (1-x)^{1-x}$

The steady state world aggregate stock of capital and aggregate consumption are given by:

$$\frac{k^*}{2} = \frac{\tilde{\gamma}}{1-\tilde{\gamma}} \left(\frac{\Psi}{\rho + \delta} \right)^{\frac{1}{1-\tilde{\gamma}}} \tag{23}$$

$$\frac{c^*}{2} = \left(\frac{\rho + \delta(1-\tilde{\gamma})}{1-\tilde{\gamma}} \right) \left(\frac{\Psi}{\rho + \delta} \right)^{\frac{1}{1-\tilde{\gamma}}} \tag{24}$$

$$s^* = \tilde{\gamma} \frac{\delta}{\rho + \delta}$$

where c is the world per capita consumption: $c = c^N + c^S$ and s^* is the steady state savings rate. It is interesting to note that the aggregate steady state is the addition of the steady states that the countries have in autarky. However, with trade, each country's steady state is different to the autarky steady state. The following proposition characterizes the country specific steady state levels of capital and consumption.

Proposition 1 *Given an initial wealth distribution there exists a unique steady state level for the world aggregate variables. However, the steady state level for the country aggregates is a function of the initial wealth distribution, hence there are an infinite number of such steady states.*

Proof. From (23) and (24) we can verify that the steady state levels for the world aggregate variables are independent of (16). With the solution to c^* given in (24) we can solve for c^{N*}

and c^{S*} using (16)

$$c^{N*} = \frac{1}{1+\varpi}c^* \quad (25)$$

$$c^{S*} = \frac{\varpi}{1+\varpi}c^* \quad (26)$$

Note that the consumption levels at the steady state are functions of the initial conditions for capital:

$$k^{N*} = \frac{1}{\rho}\left(\frac{1}{1+\varpi}c^* - w^*\right) \quad (27)$$

$$k^{S*} = \frac{1}{\rho}\left(\frac{\varpi}{1+\varpi}c^* - w^*\right) \quad (28)$$

Note that the steady state levels for the country aggregates are a function of the initial wealth distribution, hence they are not unique. ■

This is the result of no convergence in income levels presented in Bajona and Kehoe (2006) and Chen (1992). Only in the particular case in which $\varpi = 1$ $\left(\text{this means when } k^N(0) = k^S(0)\right)$ both economies converge to the same steady state. It is easy to show that the steady state will be the same for both economies and the same as the one in autarky, simply evaluate (27) and (28) at $\varpi = 1$. It is also worth emphasizing that if $\varpi \neq 1$, then one country will have a larger steady state than the other, and in particular, one country will have a larger steady state than in autarky and the other a smaller steady state than in autarky.

3.2 Dynamics

The system of differential equations (17) , (18) and (19) govern the transition of these economies. If we linearize the system in the neighborhood of the steady state and analyze the resulting characteristic equation we are able to solve for the eigenvalues[11]. There is one negative eigenvalue, therefore the system is saddle path stable. Moreover, the stable arm corresponds to the eigenvector associated to the negative eigenvalue and is a ray that goes through the steady state. The negative root is independent of ϖ and is the same eigenvalue as in autarky. However, the stable arm depends on ϖ and there are an infinite number of them. Given an initial

[11] The linearized system together with the eigenvalues for all possible cases are presented in the Technical Appendix which is available upon request.

condition, we have a unique ϖ and stable arm that leads us to the steady state. Locally, we know that the system is stable. Moreover, note that once we are inside the FPE set, and stay inside, the world behaves as a standard Ramsey-Cass-Koopmans neoclassical growth model which we know is globally stable. I will first argue that if initial conditions belong to region A (A'), then both countries stay inside the cone and converge to the steady state globally. Then, if initial conditions belong to region B (B') countries could leave the FPE set in finite time. In order to show this, we will first consider initial conditions on the boundary of the FPE set and evaluate the direction in which relative factor supplies of each country move. Then, I will focus my attention inside the FPE set.

From the previous equilibrium conditions it is clear that if the capital stocks in each country grow at the same rate as the world aggregate stock, then we have that FPE will hold in every given period. This is because the boundaries of the FPE set grow also at the same rate. However, the growth rate of capital of the country that starts with a higher stock could be larger or smaller than the growth rate of the aggregate stock of capital, and the growth rate of the country with initial stock that is lower could have a larger or smaller growth rate. Note that since $k(t) = k^S(t) + k^N(t)$ then $\dot{k}(t) = \dot{k}^S(t) + \dot{k}^N(t)$, so if $\dot{k}^S(t) > \dot{k}(t)$ then $\dot{k}^N(t) < \dot{k}(t)$.

This can be seen in the following way, let $z^i(t) = k^i(t)/k(t)$, then differentiating with respect to time we get:

$$
\begin{aligned}
\frac{\dot{z}^i(t)}{z^i(t)} &= \frac{\dot{k}^i(t)}{k^i(t)} - \frac{\dot{k}(t)}{k(t)} = g_{k^i} - g_k \\
\frac{\dot{z}^N(t)}{z^N(t)} &= (w(t) - \rho\hat{w}(t)) \left[\frac{k^S(t) - k^N(t)}{k^N(t)\,k(t)}\right] \\
\frac{\dot{z}^S(t)}{z^S(t)} &= (w(t) - \rho\hat{w}(t)) \left[\frac{k^N(t) - k^S(t)}{k^S(t)\,k(t)}\right]
\end{aligned}
$$

where $\hat{w}(t)$ is the present discounted value of wage income[12]. We can have different situations

[12] $\hat{w}^i(t) \equiv \int_t^\infty G^i_{l^i}\left(k^i(s);p\right) e^{-\left(R^i(s)-\delta\right)s} ds.$

depending on the relative magnitude of the capital stocks and wages[13].

$$\text{If. } w(t) - \rho\hat{w}(t) \quad > 0 \Rightarrow \begin{cases} g_{K^N} - g_K < 0 \\ g_{K^S} - g_K > 0 \end{cases} \Rightarrow g_{K^N} < g_{K^S}$$

$$\text{If. } w(t) - \rho\hat{w}(t) \quad < 0 \Rightarrow \begin{cases} g_{K^N} - g_K > 0 \\ g_{K^S} - g_K < 0 \end{cases} \Rightarrow g_{K^N} > g_{K^S}$$

Suppose that at $t = 0$ we start in the boundary of the FPE set, then if $\dot{z}^S(t) > 0$ the growth rate of capital from south is larger than north and countries move strictly inside the FPE set. The next lemma will show that whenever the countries are at the boundary of the FPE set they will move away from there. If $k(0) > k^*$ the economies leave the set and move outside and if $k(0) < k^*$ the economies move strictly inside.

Lemma 1 *Suppose that when countries start trading, country's factor supplies are at the boundary of the FPE set. If* $k(0) < k^*$ *then* $\dot{z}^S(t) > 0$ *and if* $k(0) > k^*$ *then* $\dot{z}^S(t) < 0$.

Proof. Without lost of generality assume that initial conditions are such that $z^S(0)$ is at the boundary of the FPE set. In the neighborhood of the steady state $\rho\hat{w}(t) = (c^* - k^*\lambda^+ + k(t)(\lambda^+ - \rho))$ where λ^+ is the positive eigenvalue of the dynamic system inside the FPE set. At the boundary and inside the FPE set, $w(t) = (1 - \tilde{\gamma})\Theta(k(t)/2)^{\tilde{\gamma}}$. Then note that $(1 - \tilde{\gamma})\Theta(k(t)/2)^{\tilde{\gamma}} - (c^* - k^*\lambda^+ + k(t)(\lambda^+ - \rho))$ is a decreasing continuos function of $k(t)$ and it crosses zero at k^*. Hence if $k(0) < k^*$ then $w(t) - \rho\hat{w}(t) > 0$ and $\dot{z}^S(t) > 0$ and if $k(0) > k^*$ then $w(t) - \rho\hat{w}(t) < 0$ and $\dot{z}^S(t) < 0$. ■

This implies that if we are in region A (A') then the economies stay in region A (A'), however if we are in region B (B'), the economies might leave the FPE set. In particular, if they start in the boundary of the set, the ray dividing region B (B') from region C (E), they move to region C (E).

Alternatively, we can solve for the slope of the saddle path and compare this slope with the slope of the boundary of FPE set. If the slope of the saddle path is larger, then the labor intensive country is growing faster inside the FPE set than the capital intensive country. In other words, that the relative capitals, $\frac{k^S(t)}{k^N(t)}$ is increasing over time if countries start in region

[13] Note that there are in total four cases. However, since we are assuming that initially $k^N(0) > k^S(0)$ we only have two sensible cases, the ones in which $k^N(t) > k^S(t)$. Below I show that if $k^N(0) > k^S(0)$ this will imply that $k^N(t) > k^S(t)$ for all t.

A (A') and decreasing over time if countries start in region B (B').

Lemma 2 *The slope of the saddle path is larger than the slope of the lower boundary of the FPE set.*

Proof. The ratio $\dot{k}^S(t)/\dot{k}^N(t)$ is the slope of the saddle path and we denote it by $\psi'_{K^S}(k^N)$. The exact solution to the slope is

$$\lim_{t\to\infty}\frac{\dot{k}^S(t)}{\dot{k}^N(t)} = \psi'_{k^S}\left(k^{N*}\right) = \frac{-\alpha(1-\varpi)-\varpi\tilde{\lambda}}{\alpha(1-\varpi)-\tilde{\lambda}}$$

where $\tilde{\lambda} = \left(-\rho - \sqrt{\rho^2+4\rho\alpha}\right)$ and $\alpha = (1-\tilde{\gamma})(\rho+\delta)\left(\frac{\rho+\delta(1-\tilde{\gamma})}{\tilde{\gamma}\rho}\right)$. We want to show that:

$$\psi'_{k^S}\left(k^{N*}\right) > \frac{(1-\theta_1)\tilde{\gamma}2-(1-\tilde{\gamma})\theta_1}{(1-\tilde{\gamma})\theta_1}$$

where the right hand side is the slope of the lower bound of the FPE set in the State Space II. $\bar{\varpi}$ is given by the ratio of consumptions at the steady state at the lower bound of set

$$\bar{\varpi} = \frac{K^{S*}}{K^{N*}}$$

Substituting $\tilde{\lambda}, \bar{\varpi}$ α into $\psi'_{k^S}\left(k^{N*}\right)$ we find that in order for this inequality to hold we need that $s^* = \tilde{\gamma}\delta/(\delta+\rho) < 1$ which is trivially satisfied. Therefore, the slope of the saddle path is larger than the slope of the lower boundary of the set either in State Space I and II[14]. ■

Given the previous results it might be tempting to think that south might eventually catch up to north. However, the stock of capital from the relatively capital abundant country will always be larger than the capital from the labor abundant country during the transition to the steady state and at the steady state. In other words, south will never catch up with north as long as $k^N(0) > k^S(0)$.

Proposition 2 *Suppose that countries are inside the FPE set. Then south will remain relatively more labor intensive than north, $k^N(t) > k^S(t)$, all t.*

Proof. Taking the difference between the capital stocks in both countries we find that:

$$k^N(t) - k^S(t) = \frac{1}{\rho}\left(\frac{1-\varpi}{1+\varpi}\right)c(t) > 0 \quad \text{all } t \tag{29}$$

[14] Note that the slope of the bound of the FPE set is always larger in State Space II than in State Space I.

■

The difference in the capital stocks is not constant, it changes over time and gets wider as long as consumption is growing. In the steady state, the difference is constant and a function of the initial conditions ϖ as we proved before. To see this, take the limit of (29) as $t \to \infty$ and then difference between (27) and (28) is equal to $\frac{1}{\rho}\left(\frac{1-\varpi}{1+\varpi}\right)\left(\frac{\rho+\delta(1-\tilde{\gamma})}{1-\tilde{\gamma}}\right)\left(\frac{\Psi}{\rho+\delta}\right)^{\frac{1}{1-\tilde{\gamma}}}$.

Figure 4 presents a projection of the phase diagram on the $\left(k^N - k^S\right)$ plane and State Space I. Several exact trajectories starting in region A are presented that illustrate the previous findings. As we can see, for a given initial condition (circles), the economy converges to a steady state inside of the FPE set (stars). Also, initial conditions at the boundary of the set leave the boundary and converge to a steady state strictly inside the FPE set.

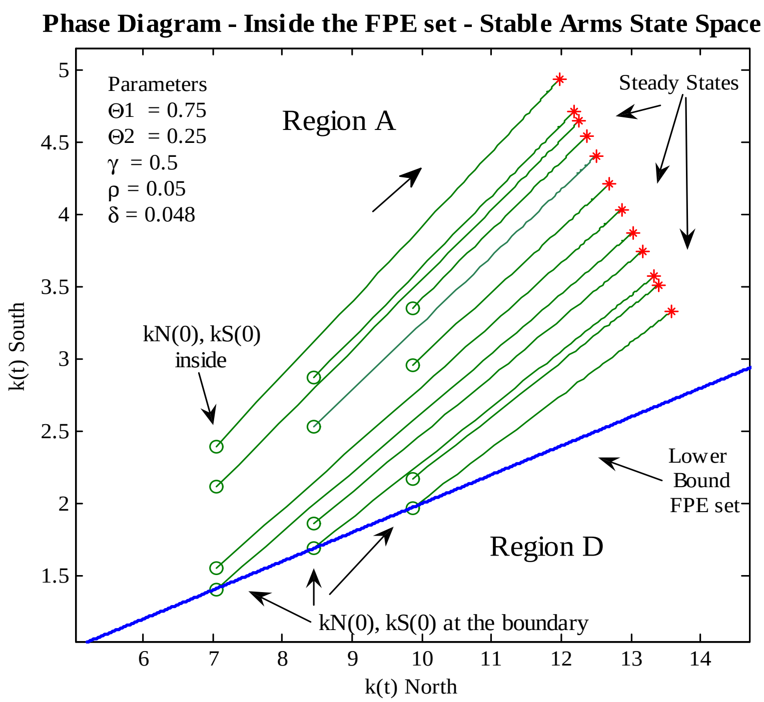

Figure 4

Figure 5 presents the same findings on State Space I. Several exact trajectories starting in region A' and with initial conditions at the boundary of the FPE set are shown to remain inside the FPE set. Therefore, if countries start trading with relative factor endowments below the steady state and belonging to the cone of diversification, then factor prices are equalized at time zero and remain equalized thereafter. Both countries diversify their production along the transition path and at the steady state. The steady state will be a function of initial conditions.

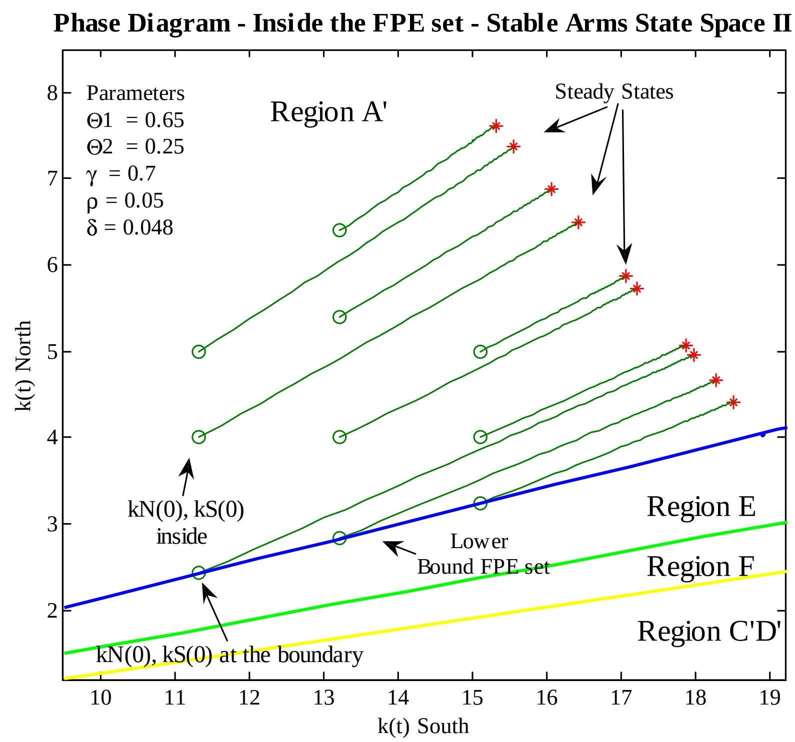

Figure 5

Figure 6 and 7 present exact trajectories starting from region B and B' respectively. These are initial conditions (circles) in which both countries start above their steady state and inside the cone of diversification. The slope at which they reach the steady state (stars) is larger than the slope of the lower boundary of FPE set as was proven in Lemma 2. Since they are converging from above we could have some trajectories leaving the FPE set. In particular, from Lemma 1 we know that trajectories that have initial condition at the lower bound of the set leave it.

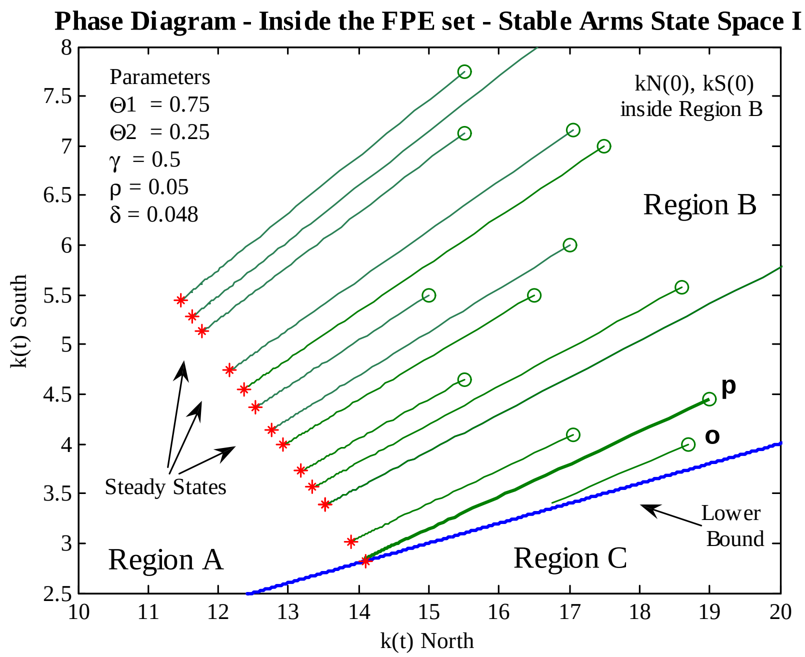

Figure 6

There is one trajectory of particular interest to us, the darker one labeled **p** in both figures 6 and 7**.** This trajectory has the property that it converges to a steady state at the lower boundary of the FPE set. This is a specialized steady state which in the case of figure 6 it has the labor intensive country specialized and in the case of figure 7 it has the capital intensive country specialized. The slope at which it reaches the steady state is given by $\psi'_{k^S}\left(k^{N*}\right)$ evaluated at $\tilde{\lambda}, \bar{\varpi}$ and α. The next Lemma will argue that any initial condition in region B (B') between **p** and the lower boundary of the FPE set must leave the set in finite time.

Phase Diagram - Inside the FPE set - Stable Arms State Space II

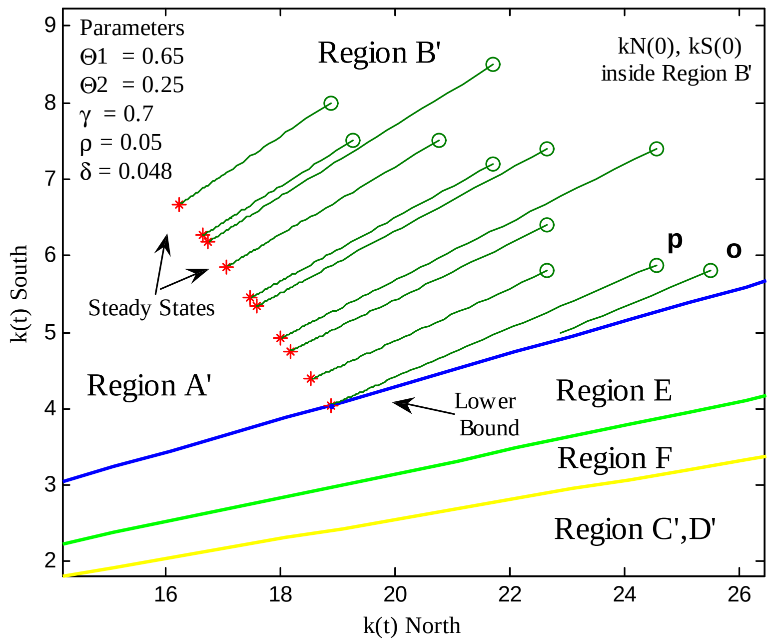

Figure 7

Lemma 3 *Trajectories with initial conditions belonging to region B (B') below (above)* **p** *leave (stay in) region B (B').*

Proof. Trajectories cannot cross each other. Since **p** is a trajectory that converges to a steady state at the lower bound of the FPE set, any trajectory below **p** must leave the set in finite time, unless they converge to the same steady state as **p**. However, from Proposition 1, every steady state reached from inside the FPE set is a function of initial conditions, therefore it cannot be the same steady state as the one **p** is heading if it always stays inside the FPE set. In this way, trajectories below **p** reach the lower bound in finite time. For instance, trajectories **o** in figure 5 and 6 have such property. From Lemma 1 we know that once the countries reach the lower bound of the FPE set in region B (B'), they leave the set. The proof for trajectories above **p** is analogous.

Therefore, if countries start trading with relative factor endowments above the steady state and belonging to the FPE set, region B (B'), then depending on initial conditions there are two scenarios. Factor prices are equalized at time zero and remain equalized thereafter in which case both countries will be diversifying their production along the transition to the steady state. This will occur if, when countries start trading, their factor endowments belong to region B (B') and above trajectory **p**. Trajectory **p** is a special and interesting case because it is a trajectory in which countries remain diversified during the transition but converge to a steady state in which at least on country is specialized. The alternative scenario is that factor prices are equalized at time zero and both countries diversify their production but countries leave the FPE set in finite time. In the case of figure 5, the labor intensive country will end up specialized, and in the case of figure 6, the capital intensive country will end up specialized. Any combination of factor supplies below trajectory **p** in the FPE set have this property[15].

■

4 No Factor Price Equalization

In this subsection we will consider cases in which relative factor supplies from each country are considerably different from each other. In particular they are different enough such that factor prices are not equal and countries might specialize in the production of the good in which they are more abundant. Initial distribution of endowments such that they belong to regions C,D, E, F, C'D' in the state space.

The next propositions, which are standard results from the static model. will show that when countries specialize there are a unique set of equilibrium prices. We will show this for the case in which south is relatively more labor abundant and specializes in the labor intensive good while north diversifies its production (region C D). The rest of the cases are analogous. The following conditions regarding relative factor supplies have to hold for this to be true:

$$\frac{k^S(0)}{k^N(0)} < \frac{\theta_2(1-\tilde{\gamma})}{\theta_2(1-\tilde{\gamma}) + 2(\tilde{\gamma}-\theta_2)} \tag{30}$$

I will assume that (30) holds. In this way, when countries start trading north will produce both of the intermediate goods and export the capital intensive good, while south will only produce and export the labor intensive intermediate good 2 and import the capital intensive

[15] Note that it is possible to find a combination of parameter values in which the economies converge to a steady state in which both are specialized.

good. The problem of the agent in south has three binding constraints which imply:

$$k_1^S = 0, l_1^S = 0, y_1^S = 0$$

First I will show that for a given supply of factors satisfying (30) there is a unique equilibrium set of prices and factor prices for each country. Factor prices will not be equal and one of the countries will specialize. From the solution of the household-firm subproblem in each country and imposing trade balance, the set of prices in the world solve the following system of equations:

$$\begin{aligned}
\phi_L^N &= \phi_K^N k^N \left(\frac{1-\tilde{\gamma}}{\tilde{\gamma}}\right) + \phi_K^S k^S \left(\frac{\gamma}{\tilde{\gamma}}\right)\left(\frac{\theta_2-\theta_1}{\theta_2}\right) && (31)\\
\phi_L^S &= \phi_K^S k^S \left(\frac{1-\theta_2}{\theta_2}\right) && (32)\\
p_1 &= \frac{\left(\phi_K^N\right)^{\theta_1}\left(\phi_L^N\right)^{1-\theta_1}}{\tilde{\Gamma}\left(\theta_1\right)} && (33)\\
p_2 &= \frac{\left(\phi_K^N\right)^{\theta_2}\left(\phi_L^N\right)^{1-\theta_2}}{\tilde{\Gamma}\left(\theta_2\right)} = \frac{\left(\phi_K^S\right)^{\theta_2}\left(\phi_L^S\right)^{1-\theta_2}}{\tilde{\Gamma}\left(\theta_2\right)} && (34)\\
1 &= \frac{p_1^{\gamma} p_2^{1-\gamma}}{\tilde{\Gamma}\left(\gamma\right)} && (35)
\end{aligned}$$

We can express all of the equilibrium prices as a function of one of the prices, say wages in north (ϕ_L^N). Then prices are determined uniquely by solving:

$$\begin{aligned}
\phi_K^N &= \Psi^{\frac{1}{\tilde{\gamma}}}\left(\phi_L^N\right)^{\frac{\tilde{\gamma}-1}{\tilde{\gamma}}} && (36)\\
\phi_K^S &= \frac{\theta_2}{\tilde{\Gamma}\left(\theta_2\right)}\Psi^{\frac{\theta_2}{\tilde{\gamma}}}\left(\phi_L^N\right)^{\frac{\tilde{\gamma}-\theta_2}{\tilde{\gamma}}}\left(k^S\right)^{\theta_2-1} && (37)\\
p_1 &= \frac{\Psi^{\frac{\theta_1}{\tilde{\gamma}}}}{\tilde{\Gamma}\left(\theta_1\right)}\left(\phi_L^N\right)^{\frac{\tilde{\gamma}-\theta_1}{\tilde{\gamma}}} && (38)\\
p_2 &= \frac{\Psi^{\frac{\theta_2}{\tilde{\gamma}}}}{\tilde{\Gamma}\left(\theta_2\right)}\left(\phi_L^N\right)^{\frac{\tilde{\gamma}-\theta_2}{\tilde{\gamma}}} && (39)\\
\phi_L^S &= \frac{\left(1-\theta_2\right)}{\tilde{\Gamma}\left(\theta_2\right)}\Psi^{\frac{\theta_2}{\tilde{\gamma}}}\left(\phi_L^N\right)^{\frac{\tilde{\gamma}-\theta_2}{\tilde{\gamma}}}\left(k^S\right)^{\theta_2} && (40)\\
\left(\phi_L^N\right)^{\frac{1}{\tilde{\gamma}}} &= k^N\left(\frac{1-\tilde{\gamma}}{\tilde{\gamma}}\right)\Psi^{\frac{1}{\tilde{\gamma}}} - \Psi^{\frac{\theta_2}{\tilde{\gamma}}}\left(\phi_L^N\right)^{\frac{1-\theta_2}{\tilde{\gamma}}}\left(k^S\right)^{\theta_2}\left(\frac{\gamma}{\tilde{\gamma}}\right)\left(\frac{\theta_1-\theta_2}{\tilde{\Gamma}\left(\theta_2\right)}\right) && (41)
\end{aligned}$$

At each t, given a supply of factors in the world $\left(k^N, k^S\right)$ we can use these system to solve for the six unknown prices, $\{\phi_K^N, \phi_K^S, \phi_L^N, \phi_L^S, p_1, p_2\}$. In particular, if we show that there is a unique $\phi_L^N > 0$ that solves (41), then with ϕ_L^N together with $\left(k^N, k^S\right)$ we can solve for the rest of the prices using $(36-40)$.

Proposition 3 *For any positive $k^N, k^S \in$ Regions C,D there exists a unique $\phi_L^N > 0$ that satisfies* (41).

The proof is straightforward so I omit it. Note that these are simply two continuos functions crossing only once. The same result holds for the case in which factor supplies belong to region C'D', E and F.

We will now consider a particular case in which factor endowments belong to the boundary of the FPE set. At the boundary south will still be specialized in the production of the labor intensive good, but factor prices will be equal.

Proposition 4 *If factor endowments are at the boundary of the FPE set (initial conditions on the ray dividing regions A and B from regions C and D in figure 1), south produces only the labor intensive good and factor prices are equal.*

Proof. South at the bound implies that $k^S = \frac{\theta_2}{1-\theta_2}\frac{1-\tilde{\gamma}}{\tilde{\gamma}}\frac{k^S+k^N}{2}$. We will show that if this holds then the system of equations $(36-41)$ has a unique solution with FPE. By Proposition 3 there is a unique set of equilibrium prices that solve the system of equations. Hence, if we find a set of prices that solve the system of equations $(36-41)$ and those prices satisfy FPE these will be the only equilibrium prices. Therefore, we test if prices that satisfy FPE solve the equations. Suppose factor prices are equal, then note that (32) implies that $\frac{\phi_L}{\phi_K} = k^S\frac{1-\theta_2}{\theta_2} = \frac{1-\tilde{\gamma}}{\tilde{\gamma}}\frac{k^S+k^N}{2}$. Then from (31) using (32):

$$\frac{\phi_L}{\phi_K} = \left(\frac{1-\tilde{\gamma}}{\tilde{\gamma}}\right) k^N \frac{\tilde{\gamma}\left(1-\theta_2\right)}{\tilde{\gamma}\left(1-\theta_2\right) + \left(\tilde{\gamma}-\theta_2\right)}$$

But note that $k^S = \frac{\theta_2}{1-\theta_2}\frac{1-\tilde{\gamma}}{\tilde{\gamma}}\frac{k^S+k^N}{2}$ implies that $\frac{k^S+k^N}{2} = k^N\left(\frac{\tilde{\gamma}(1-\theta_2)}{\tilde{\gamma}(1-\theta_2)+(\tilde{\gamma}-\theta_2)}\right)$, therefore factor prices being equal is an equilibrium if factor supplies belong to the FPE set. ■

An analogous proposition to Proposition 4 can be probed for the case in which factor endowments belong to the ray dividing regions A' and B' from regions E in figure 2. In that case, north produces only the capital intensive good, south diversifies and factor prices are equal.

We will now focus on the dynamics of the model when countries start trading outside the FPE set.

4.1 Dynamics

The laws of motion governing transition for both countries are:

$$\dot{c}^N = \left(G_{K^N}\left(k^N;p\right) - (\rho+\delta)\right)c^N \tag{42}$$

$$\dot{c}^S = \left(G_{k^S}\left(k^S;p\right) - (\rho+\delta)\right)c^S \tag{43}$$

$$\dot{k}^N = \left(G_{k^N}\left(k^N;p\right) - \delta\right)k^N + G_{L^N}\left(k^N;p\right) - c^N \tag{44}$$

$$\dot{k}^S = \left(G_{k^S}\left(k^S;p\right) - \delta\right)k^S + G_{L^S}\left(k^S;p\right) - c^S \tag{45}$$

When countries start trading with factor supplies considerably different from each other factor prices will not be equal. However, will it be the case that FPE might hold in the future? If so, is it during the transition or at the steady state? In order to answer these questions we can not rule out the possibility that during the transition there might be cases in which $G_{K^N} \neq G_{K^S}$ and $G_{L^N} \neq G_{L^S}$, and cases in which they are equal. In this case, there is no closed form solution for prices as was the case when countries were diversifying their production. However, we can solve for world prices at each t, using $(36-41)$ for the case in which factor supplies belong to C, D or C'D'[16].

4.2 Specialized steady states

To determine how the economies behave after they start trading outside of the FPE set, the system $(42-45)$ has to be analyzed. Before considering the dynamics let us determine the steady state of the system. From the laws of motion of the state and co-state variables, we can solve for the steady state of this economy $\{r^{N*}, r^{S*}, w^{N*}, w^{S*}, k^{N*}, k^{S*}, c^{N*}, c^{S*}\}$. Note that solving for a steady state of this system is solving for two type of steady states, one in which south is specialized and north is diversified, hence we will label this steady state as the

[16] Analogous conditions to solve for prices can be found for the case in which factor endowments belong to regions E and F. Oslington and Towers (2009) describe the conditions for a Cobb Douglas case like the one in this paper.

S-specialized steady state, and one in which north is specialized and south is diversified[17]:

Definition 1 (S-specialized steady state)

$$k^{N*} = \left(\frac{\tilde{\gamma}(1-\theta_2)+\gamma(\theta_1-\theta_2)}{(1-\tilde{\gamma})(1-\theta_2)}\right)\left(\frac{\Psi}{\rho+\delta}\right)^{\frac{1}{1-\tilde{\gamma}}} \tag{46}$$

$$k^{S*} = \left(\frac{\theta_2}{1-\theta_2}\right)\left(\frac{\Psi}{\rho+\delta}\right)^{\frac{1}{1-\tilde{\gamma}}} \tag{47}$$

Definition 2 (N-specialized steady state)

$$k^{N*} = \left(\frac{\theta_1}{1-\theta_1}\right)\left(\frac{\Psi}{\rho+\delta}\right)^{\frac{1}{1-\tilde{\gamma}}} \tag{48}$$

$$k^{S*} = \left(\frac{\tilde{\gamma}(1-\theta_1)-(1-\gamma)(\theta_1-\theta_2)}{(1-\tilde{\gamma})(1-\theta_1)}\right)\left(\frac{\Psi}{\rho+\delta}\right)^{\frac{1}{1-\tilde{\gamma}}} \tag{49}$$

Note that in both steady states, the value of aggregate consumption and capital are the same as before, (23) and (24). Another important observation is that the steady states do not depend on initial conditions, this is a key difference compared to the case we discussed before in which the steady state is a function of the initial conditions. In figure 2, the S-specialized steady state is given by the intersection of regions A, B, C and D. In figure 3 the N-specialized steady state is given by the intersection of regions A', B' and E. Note that if initial conditions are such that they belong to State Space I then either countries converge to a steady state in the FPE set or the S-specialized steady, while if initial conditions belong to State Space II then countries converge to a steady state inside the FPE set or the N-specialized steady state.

In order to understand what happens in the transition we need to evaluate the stability of the system of differential equations that describes the evolution of the economy. By linearizing the system in the neighborhood of the steady state one can show that the system is saddle path stable. There are two negative roots and the eigenvectors associated with these roots

[17] There can also be a steady state in which both countries specialize, however unless we assume that $\frac{\gamma}{1-\gamma} = \frac{L^N}{L^S}\frac{1-\theta_2}{1-\theta_1}$ it is strictly inside the FPE set, therefore countries will never reach that steady state. The stady state in which both countries are specialized is given by $k^{N*} = \left(\frac{((1-\gamma)\theta_2)^{(1-\gamma)\theta_2}(\theta_1\gamma)^{1-(1-\gamma)\theta_2}}{\rho+\delta}\right)^{\frac{1}{1-\tilde{\gamma}}}$ and $k^{N*} = \left(\frac{((1-\gamma)\theta_2)^{1-\theta_1\gamma}(\theta_1\gamma)^{\theta_1\gamma}}{\rho+\delta}\right)^{\frac{1}{1-\tilde{\gamma}}}$. Note that by assuming that set of parameters, the N-specialized and the S-specialized steady states will be equal.

determine the slope at which the trajectories approach the steady state. To understand this, the eigenvectors are tangent to the stable manifolds of the nonlinear system at the steady state, and the eigenvalues are used to distinguish stable from unstable manifolds. Stable manifolds are the ones associated with the negative eigenvalues. Let us label the roots in ascending order as ψ_1, ψ_2, ψ_3 and ψ_4 where the first two are the stable ones. The solution to the differential equations takes the form $y(t) = \sum_i^2 \omega_i e^{\psi_i t}$ where ω_i is determined from boundary conditions. There are two paths that can lead us to the steady state and I label them pike and backroad[18]. Pike is the one associated with the less negative root (ψ_2) and backroad with the most negative root (ψ_1). Convergence through the pike is slower than through the backroad and the path of adjustment of capital stocks are toward the pike since as $t \to \infty$ the eigenvalue that dominates the system is ψ_2. Note that the solution to the differential equations are linear combinations of both where the relative weight placed on each of them depends on initial conditions. In State Space I, figure 2, the backroad is the ray dividing region C from region D.

We can solve for the slope of the pike as a function of the eigenvalue and it is given by the following expression:

$$S(\psi_2) = \frac{(\psi_2)^2 - f\psi_2 + a}{\psi_2 g - b}$$

where a, b, f and g are constants from the quartic equation (50)[19]. As opposed to the cases considered inside the FPE set, the slope of the saddle path is unique and is not a function of initial conditions. Another important observation is that all trajectories starting outside the FPE set and heading to the steady state should reach the steady state with this slope. Trajectories heading through the backroad need to start on the backroad to converge to the steady state, otherwise they will take the pike. Note that it can happen that trajectories heading to the steady state can reach the FPE set before they reach the pike. This could happen if the lower boundary of the set is reached. Once the countries reach the FPE set, depending on wether they reach region A or B (A' or B'), we know that we converge strictly inside the FPE set or return back to a region outside of the FPE set. In light of this, it becomes important to compare the slope of the lower bound of the FPE set and the slope

[18] The names come from Stokey (1998). Pike is the slowest trajectory while Backroad is the fastest one.

[19] Note that the slope of the pike is different in regions C, D compare to region E. The eigenvalues and the parameters of the quartic equation are different in both cases. We find that in both cases there are two negative eigenvalues. Therefore, the arguments we are making here hold for regions C, D and E.

of the pike. If the slope of the pike is lower than the slope of set, then depending on initial conditions the FPE set might be reached in finite time or not. For instance, if the economies start trading in region D the FPE set is reached in finite time. After countries reach the FPE set, the system governing the transition is the one inside of the FPE set (CASE 1 above region A) and countries converge to a steady state strictly inside the set. On the other hand, if the economies start trading in region C, then the FPE set will be reached only at the steady state. These findings have different implications over the pattern of specialization and factor prices. We will prove that the slope of the pike is lower than the FPE set.

In order to prove this, it is necessary to compute the eigenvectors associated with the negative eigenvalues, in particular the slope of the pike and compare it to the slope of the FPE set. Given the restriction over parameters in the model, we will show that it is not possible for the slope of the pike to be higher than the slope of the FPE set. If this is the case, then from region D the FPE set is reached in finite time and if the initial conditions belong to region C, then the economies converge to a steady state in which factor prices equalize, but on the boundary of the FPE set, corresponding to the S-specialized steady state[20].

Proposition 5 *The slope of the pike is lower than the slope of the lower bound of the FPE set.*

Proof. Let us label the roots in ascending order as ψ_1, ψ_2, ψ_3 and ψ_4 where the first two are the stable ones. In order to solve for these roots it is necessary to solve a quartic equation. In particular note that the quartic is given by:

$$Q(x) = x^4 - (f+j)\,x^3 + (a+d-gh+fj)\,x^2 + (bh+cg-df-aj)\,x + (ad-bc) \tag{50}$$

where the constants a, b, c, d, f, g, h, j come from the fundamental equation and are found by linearizing the system of four differential equations around the S-specialized steady state. Note that $\lim_{x\to\infty} Q(x) = \infty$ and $\lim_{x\to-\infty} Q(x) = \infty$. Then for $x \in (\psi_2, \psi_3)$, $Q(x) > 0$. The slope of the lower boundary of the FPE set is given by:

$$\kappa \equiv \frac{\theta_2(1-\tilde{\gamma})}{\theta_2(1-\tilde{\gamma}) + 2(\tilde{\gamma}-\theta_2)}$$

[20] We prove this result for the case in which countries approach the steady state from region C, D. The case in which countries reach the steady state from region E is analogous.

We want to show that

$$S(\psi_2) < \kappa$$

First note that the slope of the pike is increasing in ψ_2

$$S'(\psi_2) = \frac{(\psi_2)^2 g + (b-a) g}{(\psi_2 g - b)^2} > 0$$

since $a < 0$, $g > 0$ and $b > 0$ (this is not hard to show) $S'(\psi_2) > 0$. Now, let ψ^* be the value such that $S(\psi^*) = \kappa$, note that if $\psi^* > 0$ then we are done since the slope is increasing in ψ and hence $S(\psi_2) < S(\psi^*) = \kappa$. However, ψ^* takes two values and one is $\psi^* < 0$. Note that $Q(\psi^*) > 0$. But this implies that either $\psi^* < \psi_1$ or $\psi^* > \psi_2$. The quartic has three inflexion points that solve $Q(x)' = 0$. Note that $Q(x)' = 4x^3 - 3(f+j)x^2 + 2(a+d-gh+fj)x + (bh + cg - df - aj)$ and by Vieta's theorem the product of the roots of the cubic have the sign of -$(bh + cg - df - aj)/4$ which is negative since $h > 0$, $f > 0$, $j > 0$, $d < 0$, and $c > 0$. Therefore either all the roots are negative, which cannot be the case since two roots of the quartic are positive, or only one root is negative. This means that there is only one inflexion and it is located between ψ_1 and ψ_2. Hence, since $Q(\psi^*)' > 0$ then $S(\psi_2) < S(\psi^*) = \kappa$ ■

Since the slope of the pike is lower than the slope of the lower bound of the FPE set and since both lines cross at the steady state, then trajectories heading to the steady state from region D will reach the lower bound of the FPE set before they reach the pike, therefore suggesting that they cross to region A.

Figure 8 presents these findings. The dash rays are the slope of the Pike and the slope of the Backroad. The darker ray is the lower boundary of the FPE set. As you can see, the Pike and the lower boundary of the FPE set cross at the S specialized steady state. For values of relative factor supplies below the steady state, the Pike belongs to region A and for values of relative factor supplies above the steady state the Pike belongs to region C. This is because the slope of the Pike is lower than the slope of the lower boundary of the FPE set as was shown in Proposition 5. Several exact trajectories are presented in the figure as well. It is evident that trajectories that start in region D cannot cross the Backroad and will reach the lower boundary of the cone of diversification in finite time. After they reach the lower boundary of the FPE set, from Lemma 1 we know that the economies converge strictly inside of the set, hence region A. The economies will converge to a steady state strictly inside of the FPE set and this steady state will be a function of initial conditions. In particular, it will be a function of the relative factor supplies between the countries that they had at the moment they reached the lower boundary of the set. The theory predicts that if at the timing in which the economies open to trade, and provided the factor supplies belong to region D, south will be specialized in the labor intensive good while north will be diversifying. The reason is that south has a cost disadvantage to produce the capital intensive good and it is optimal to import these goods from north and employ all of its factors in the labor intensive industry. However, the dynamics of the model predict that south will start producing the capital intensive good, overcoming the cost disadvantage by accumulating enough capital during the transition. If there were restrictions from opening the capital sector at south, then the economies will converge to the S-specialized steady state, but since there are no restrictions they will converge to a steady state strictly inside the FPE set.

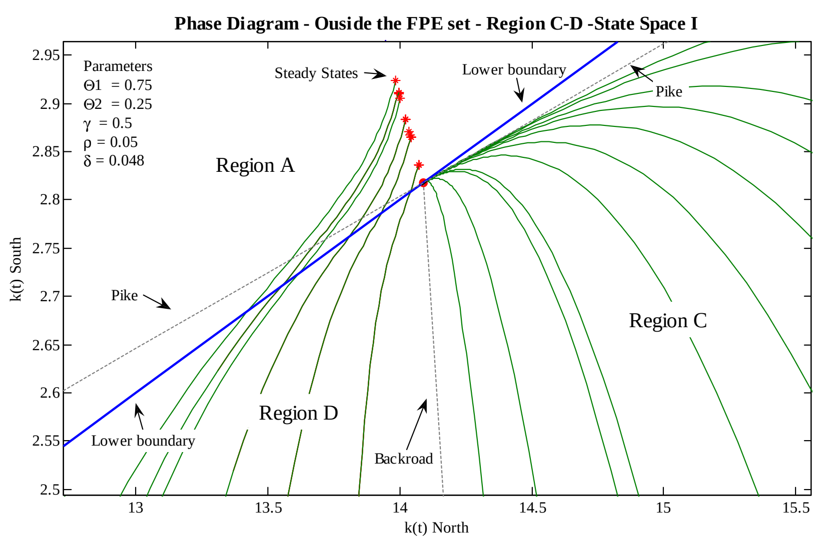

Figure 8

Now lets consider what happens with trajectories starting in region C. As was shown, trajectories are converging to the S-specialized steady state through the Pike. Trajectories that start below the Pike will head towards it from below as it is shown in the picture. These trajectories will not cross the FPE set since the Pike is reached before that. Factor prices will be equalized but at the S - specialized steady state. Therefore, the theory predicts that when countries start trading and factor supplies belong to region C, south will remain specialized forever. The figure presents several of those trajectories.

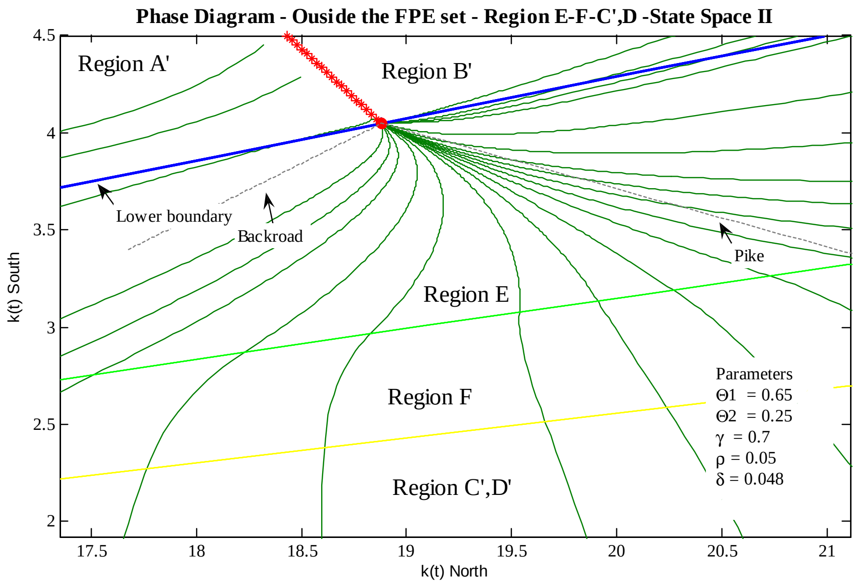

Figure 9

Figure 9 presents the same findings for State Space II. As before, the dash rays are the slope of the Pike and the slope of the Backroad. The darker ray is the lower boundary of the FPE set. As you can see, the Pike and the lower boundary of the FPE set cross at the N specialized steady state in this case. As in the case in Figure 8, it is evident that the slope of the pike is lower than the slope of the lower bound of the FPE set. In this case, the slope of the pike changes sign compared to the case before. This has considerable implications over the dynamics. For instance, in figure 8 in order to approach to the S- specialized steady state initial conditions have to be to the right of the backroad. These initial conditions implied that both countries had to be above their steady state levels. In that case also, since the slope of the pike was positive, during the transition to the steady state, south overshooted its steady state. In figure 9 this is different. Initial conditions between the backroad and the pike can be bellow the steady state and it is not south that overshoots, it is north.

Another crucial difference in the case of figure 9 is that during the transition to the steady state there are several changes in the pattern of specialization. Consider initial conditions between the backroad and the pike starting in region C',D'. Countries start trading with very different factor endowments. South has a cost advantage in the production of labor intensive goods and specializes its production while north diversifies. As countries start accumulating factors the comparative advantage changes and countries enter to region F where both countries specialize, south in labor intensive goods and north in capital intensive[21]. During the period in which countries belong to region F we can observe that south is accumulating faster capital than north. This eventually will result in south changing its comparative advantage again and both economies enter region E where south is diversified and north specialized. Then the economies move in the direction of the pike and converge to the N specialized steady state. Similar patterns will arise for trajectories starting to the left of the backroad, however, they will reach region A' in finite time. Then, factor prices will be equal and both countries are producing both goods. Eventually they reach a steady state strictly inside the FPE set.

What remains is to characterize the area between the Pike and the lower boundary of the FPE set. Initial conditions starting there will converge to the specialized steady state from above the Pike.

[21] When both countries are specialized factor prices have a closed form solution making it easy to determine the stability of the system of 4 differential equations.

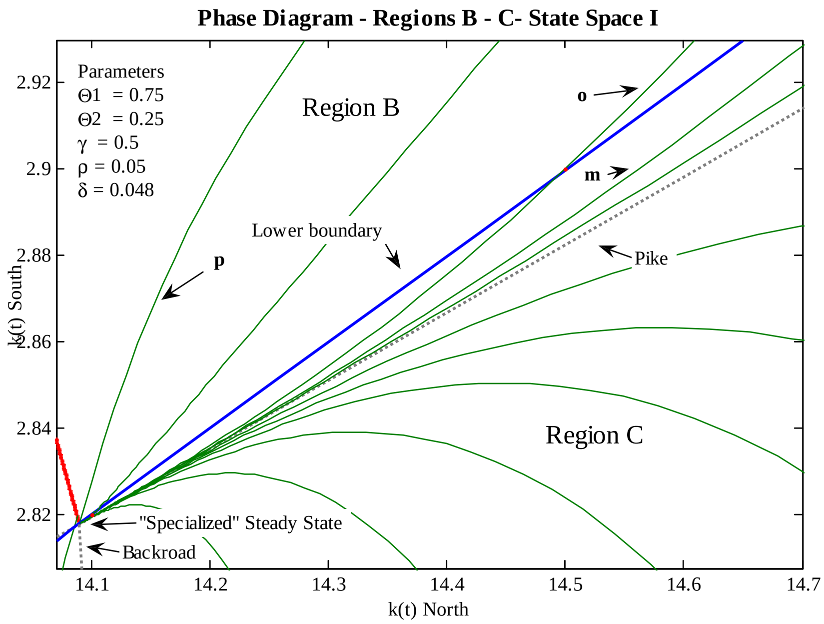

Figure 10

Figure 10 presents the behavior of the system between regions B and C[22]. Recall that the lower boundary of the FPE set separates both regions. We also present the Pike, Backroad and the lower boundary of the FPE set. In region C. the system of differential equations converges to the S specialized steady state through the Pike. In the figure we can see that trajectories that start in region C below the Pike converge from below the Pike, and trajectories that initiate between the lower boundary of the FPE set and the pike converge from above the pike. For instance, **m** is one of these trajectories. Trajectories cannot cross the pike and will not reach the lower bound of the FPE set. I also present several trajectories with initial conditions inside the FPE set, region B. For instance, **p** and **o**, which where characterized earlier, appear in the picture. Recall that **p** is the trajectory that heads to the specialized steady state from inside the region B. From Lemma 3 we know that if initial conditions are above **p** then countries stay in region B, while if they are below **p**, they reach the cone in finite time. From Lemma 1 we know that once the countries reach the lower bound of the FPE set

[22] A similar figure can be presented for the case in which countries leave region B' and move to region F. The prove is the same as the one we are going to present here that is why we omit it.

in region B, they leave the set. What is left is to characterize where these trajectories, that will leave the FPE set, are heading.

Lemma 4 *Trajectories with initial conditions belonging to the region B (B') below* $\mathbf{p}$ *leave the FPE set in finite time and converge to the specialized steady state through the pike from region C (F).*

Proof. The first part of the Lemma, that trajectories leave the FPE set, follows from Lemma 3 and Lemma 1. That trajectories converge to the specialized steady state follows from the dynamics inside region C. We know that the system is stable and that from any initial condition in region C countries converge to the specialized steady state through the Pike. ■

5 Phase Diagram

For completeness figures 11 and 12 present the Phase diagram with trajectories starting from several points of the state space. Moreover, I also consider cases in which $k^S(0) > k^N(0)$ allowing me to characterize the entire state space. The 45 degree line corresponds to the case in which $k^S(0) = k^N(0)$. As was shown before, countries remain on that ray if they start there. The red line corresponds to the set of steady states in the model. In all of them factor prices are equalized. In two steady states one of the countries is diversified, these are label by specialized steady state S-N and N-S in the figure. In the rest of the steady states, the countries diversify their production. I label things such that S-N refers to the cases in which $k^S(0) < k^N(0)$ and N-S to the case in which $k^S(0) > k^N(0)$. I also present the Backroad and the Pike for each of these two cases.

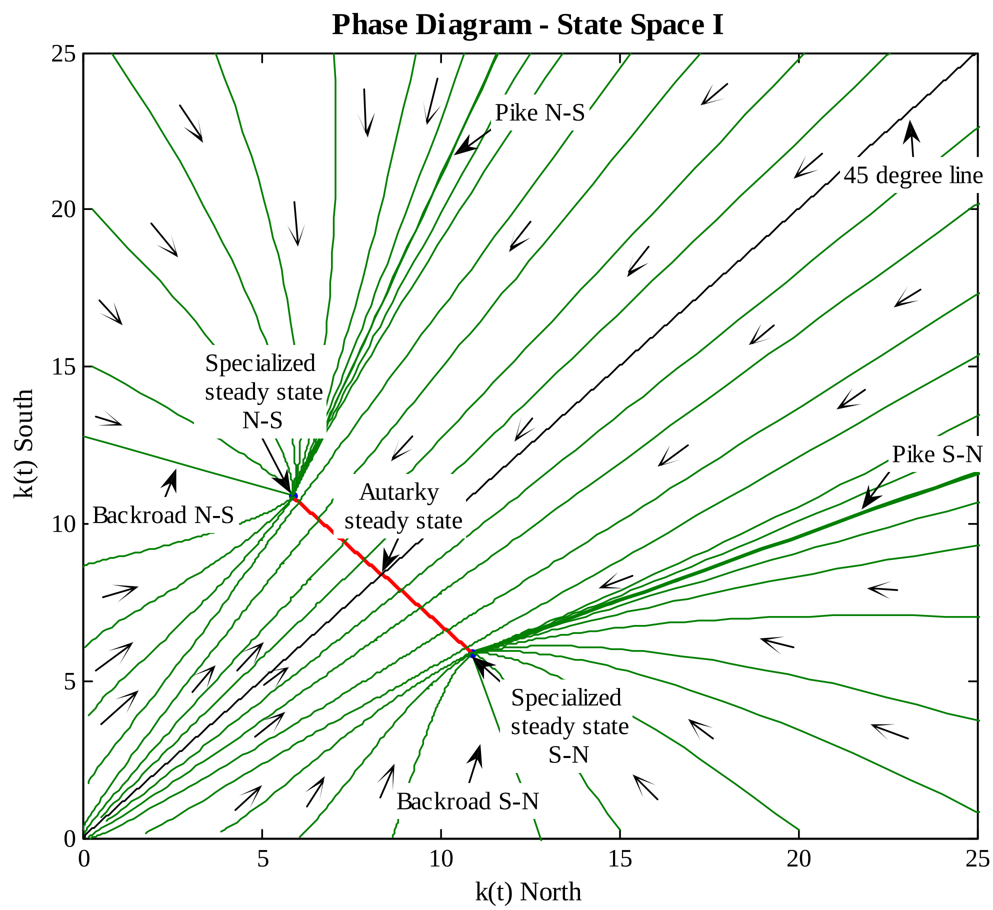

Figure 11

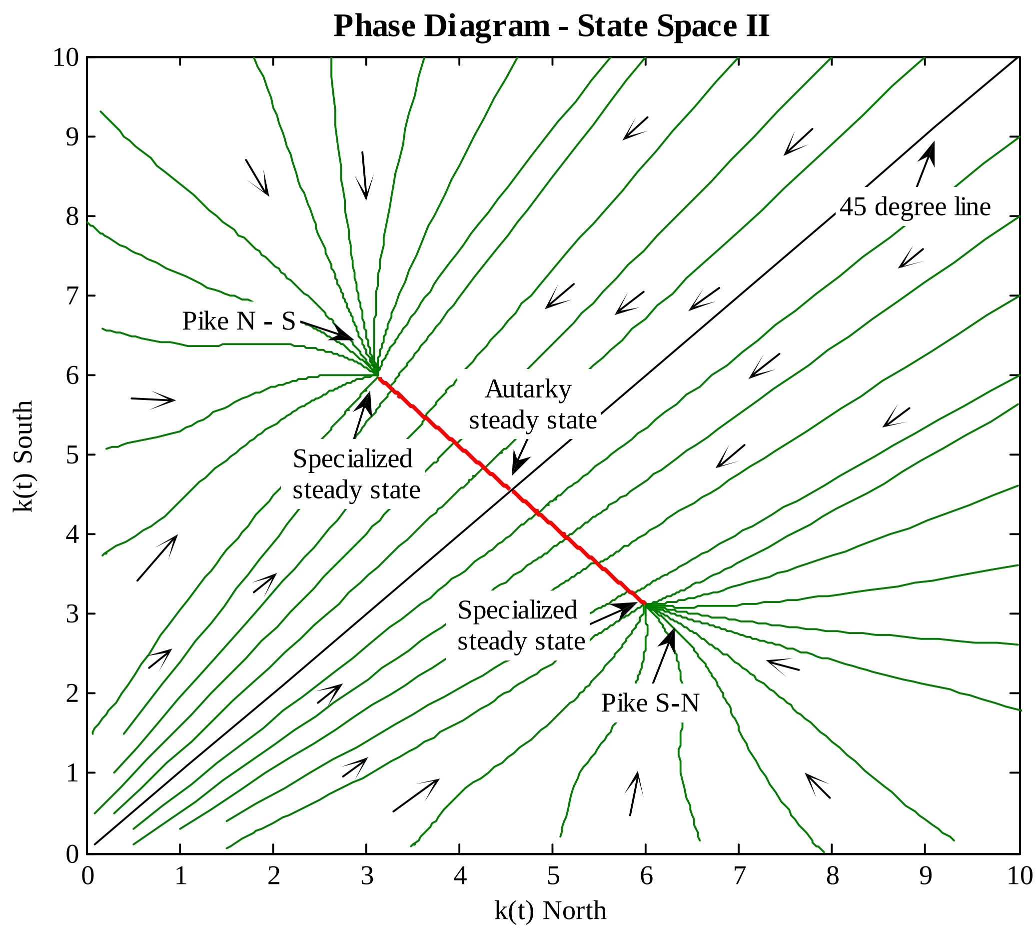

Figure 12

6 Conclusion

One of the most important theorems in international trade is the factor price equalization (FPE) theorem. It states that under certain conditions free trade in goods must lead to complete FPE. Although this hypothesis seems general, it applies to a static setup in which countries open up to trade at a given moment in time, and only if certain conditions are met do factor prices equalize. However, will this result hold in the long run? That is, does FPE at a given moment in time imply FPE forever? What are the set of factor supplies such that factor prices will be equal in a dynamic model? More important , if we observe countries trading and factor prices are not equal, will it be the case that they will eventually be equalized in finite time?

Using a standard two-(large-)country, two-factor model in which I allow free trade in two intermediate goods, I was able to characterize the set of steady states in which FPE holds. These, as we saw, depend on initial conditions of the wealth distribution across countries. I showed that for a given initial condition there is a unique steady state and characterized the restrictions on the initial conditions such that factor prices will be equalized in finite time. I showed that the system is stable and that the steady state is unique for a given initial value of the aggregate country endowments. More important, I showed that when countries start outside the FPE set they converge asymptotically to the FPE set. I were able to characterize the set of factor supplies such that the FPE set is reached in finite time.

In the static model Samuelson predicted that if countries have factor supplies not so different from each other (such that they belong to the FPE set), then factor prices will be equalized. I found that neither starting to trade inside the FPE set nor starting to trade outside the FPE set can guarantee that factor prices are equal between countries in finite time. I showed that there is a set of factor supplies such that if countries start trading with factor supplies belonging to this set, then factor prices will be equalized in finite time. I also showed that there is a set of factor supplies inside the FPE set in which factor prices will diverge in finite time between the countries.

The main message of this paper is that while a small country can grow without the retarding force of a terms-of-trade deterioration, a large country could suffer a terms-of-trade deterioration and might want to "push" itself into the diversified cone where the terms-of-trade effect is favorable with incentives to accumulate capital.

Methodologically, the paper also contributes to the literature by providing closed-form solutions to the model and showing that it is tractable and stable. Several applications can be considered for this setup. For instance, the model can be reinterpreted as a closed economy with two agents with different initial wealth distributions. One could characterize how the distribution of income evolves over time and how it might be affected by different policies (consumption tax, labor tax or income tax). Alternatively, one could evaluate the implications that opening to trade could have for the wage premium over time. This could be done by relabeling capital as skilled labor and labor as raw labor, and then rental prices would be the corresponding wages. In the model it is possible to rationalize why we could apparently observe inequality increasing in both countries during the process of development.

Another possible extension could be to incorporate start-up and closing costs to different industries. In the model, there are no costs associated with starting a new industry or closing an entire sector of production. Introducing these costs would have effects on the development of the economies, the terms of trade, factor prices and the long-run equilibrium. Also, one could apply the model to understand the role of trade on structural transformation. As we saw, the timing with which countries start trading can have a considerable impact on the pattern of specialization.

References

[1] Daron Acemoglu and Jaume Ventura, 2002. "The World Income Distribution," The Quarterly Journal of Economics, MIT Press, vol. 117(2), pages 659-694

[2] Atkeson, Andrew and Patrick J. Kehoe (2000). "Paths of Development for Early- and Late Bloomers in a Dynamic Heckscher-Ohlin Model". Federal Reserve Bank of Minneapolis Staff Report 256.

[3] Backus, David K.; Kehoe, Patrick J.; and Kydland, Finn E. 1994. "Dynamics of the trade balance and the terms of trade: The J-curve?" American Economic Review 84: 84-103.

[4] Bhagwati, Jagdish. 1958. "Immiserizing Growth: A Geometrical Note," Review of Economic Studies 25, (June), pp. 201-205.

[5] Bajona, Claustre.; Kehoe, Timothy (2006a) "Trade, Growth, and Convergence in a Dynamic Heckscher-Ohlin Model," Staff Report 378, Federal Reserve Bank of Minneapolis.

[6] Bajona, Claustre.; Kehoe, Timothy (2006b). "Demographics in Dynamic Heckscher-Ohlin Models: Overlapping Generations Versus Infinitely Lived Consumers," NBER Working Papers 12566, National Bureau of Economic Research

[7] Baldwin, Richard E. 1992 "Measurable Dynamic Gains from Trade." Journal of Political Economy 100: 162-74.

[8] Bardhan, Pranab K. 1965a. "Equilibrium Growth in the International Economy." Quarterly Journal of Economics 79, 455–464.

[9] Bardhan, Pranab K. 1965b. "International Differences in Production Functions, Trade and Factor Prices." Economic Journal 75, 81–87.

[10] Bardhan, Pranab K. 1966. "On Factor Accumulation and the Pattern of International Specialisation." Review of Economics Studies 33, 39–44.

[11] Benhabib, J., and K. Nishimura 1985 'Competitive equilibrium cycle.' Journal of Economic Theory 35, 284-306

[12] Chen Z. 1992, "Long-Run Equilibria in a Dynamic Heckscher-Ohlin Model," Canadian Journal of Economics 25: 923-943.

[13] Cuñat A., 2000. Can international trade equalize factor prices? Working paper 180. IGIER.

[14] Cuñat A, and M. Maffezzoli 2004, "Neoclassical Growth and Commodity Trade," Review of Economic Dynamics 7: 707–736.

[15] Davis, D., Weinstein, D., 2001. An account of global factor trade. American Economic Review 91, 1423–1453.

[16] Davis, D., Weinstein, D., 2003. The factor content of trade. In: Choi, E., Harrigan, J. (Eds.), Handbook of International Trade. Blackwell, New York. In press.

[17] Deardorff Alan V. 1973. "The Gains from Trade in and out of Steady-State Growth." Oxford Economic Papers 25, 173–191.

[18] Deardorff Alan V. 1994: "The Possibility of Factor Price Equalization, Revisited," Journal of International Economics, 36, 167-175.

[19] Debaere, P., Demiroglu, U., 2003. On the similarity of country endowments and factor price equalization. Journal of International Economics 59, 101–136.

[20] Dixit A. and V. Norman 1980, The Theory of International Trade: A Dual General Equilibrium Approach, Cambridge University Press.

[21] Eaton, J., 1987. A dynamic specific factors model of international trade. Review of Economic Studies 54, 325–338.

[22] Findlay, Ronald 1984. "Growth and Development in Trade Models." In Jones and Kenen (1984).

[23] Gaitan, B. and T. Roe 2007 Path Interdependence among Early and Late Bloomers in a Dynamic Heckscher-Ohlin Model, Economic Development Center Bulletin No 07-1

[24] Grossman, G. M. and E. Helpman (1991). Innovation and Growth in the Global Economy Massachusetts, MIT Press.

[25] Jensen Bjarne and C. Wang 1997 General Equilibrium Dynamics of Basic Trade Models for Growing Economies, 77-126, in (8).

[26] Johnson, Harry G. 1955. "Economic Expansion and International Trade," Manchester School 23, pp. 95-112

[27] Leamer, E. E. (1987). “Paths of Development in the Three-Factor, N-Good General Equilibrium Model.” Journal of Political Economy 95(5): 961-99.

[28] Mountford, A., 1997, Trade, Convergence and Overtaking, Journal of International Economics, 46, pp. 167-182.

[29] Oniki, H., Uzawa, H., 1965. Patterns of trade and investment in a dynamic model of international trade. Review of Economic Studies 32, 15–38.

[30] Oslington, Paul and Towers, Isaac 2009 "Trade, Migration and Inequality in a World without Factor Price Equalisation" Review of International Economics (Forthcoming)

[31] Schott, P. K. (2003). “One Size Fits All? Heckscher-Ohlin Specialization in Global Production.” American Economic Review 93(3): 686-708.

[32] Smith M. Alasdair M. 1976. “Trade, Growth and Consumption in Alternative Models of Capital Accumulation.” Journal of International Economic 6, 371–384

[33] Smith M. Alasdair M.1977. “Capital Accumulation in the Open Two-Sector Economy.” Economic Journal 87, 273–282. Reprinted in Bhagwati (1981, chapter 25).

[34] Stiglitz J.E. 1970, “Factor Price Equalization in a Dynamic Economy,” Journal of Political Economy 78: 456-488.

[35] Stokey Nancy 1996 "Free Trade, Factor Returns, and Factor Accumulation," Journal of Economic Growth, 1, pp. 421-447.

[36] Ventura J. 1997, “Growth and Interdependence,” Quarterly Journal of Economics 112: 57-84.