

Optimal Disclosure of Private Information to Competitors

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Abstract

I consider a duopoly model with differentiated substitutes, price competition, and uncertain demand, in which one firm has an information advantage over a competitor. I study the incentives of the informed firm to share its private information with its competitor and the incentives of a regulator to constrain or enforce disclosure in order to benefit consumers. I show that full disclosure of information is optimal for the informed firm, because it increases price correlation and surplus extraction from consumers. I also show that the regulator can increase expected consumer surplus and welfare by restricting disclosure, but that, surprisingly, consumers can benefit from the regulator privately disclosing some information to the competitor. Disclosure increases the ability of firms to extract surplus from consumers, but private disclosure allows consumers to arbitrage prices by creating a coordination failure in firm pricing. I show that private partial disclosure is optimal for consumers when firms offer sufficiently close substitutes. My findings highlight the effect of an uneven distribution of consumer data between firms on welfare allocation. They also inform an ongoing policy debate about how regulatory entities can control the dissemination of information between firms to protect consumers.

Some firms can gather more information than their competitors about market features like demand, given their size or incumbency status. For instance, online platforms like Amazon engage in massive data collection and analysis by gathering and processing information generated through trade that other sellers on the platform can't replicate. As a seller themselves, they can use this information to guide their own pricing and control the information observed by other sellers. In settings of information asymmetry, information disclosure between firms affects firm behavior and therefore also impacts consumers and welfare. The use of private information as a competitive advantage by online platforms has attracted the attention of regulatory entities in the US and Europe.¹ When there is an uneven distribution of consumer data between firms, regulatory interventions to control information disclosure can potentially redistribute surplus between firms and consumers or increase welfare.

In this paper, I study the role of information disclosure as a pricing persuasion device, through which a firm with an information advantage or a regulator can influence the pricing of a competing firm. I examine the informed firm's incentives to commit to share its private information with its competitor and the role of a regulator who commits to control information disclosure between firms to benefit consumers. Specifically, I analyze a stylized duopoly model with information asymmetry about demand and a binary state. I consider an informed firm which privately learns the level of the demand and an uninformed firm that has no private information. Demand is linear and firms face uncertainty about its level, which can be either low or high. Firms offer differentiated goods, such that consumer willingness to pay for a good depends on its substitutability with the competitor's. Firms compete by simultaneously and non-cooperatively setting prices to maximize their expected profits. In this context, I address the following questions: What is the informed firm's optimal disclosure policy as a competitor in the market? How can a regulator constrain or enforce information disclosure to benefit consumers?

I characterize the optimal disclosure for firms and consumers. I show that the welfare implications of disclosure are determined by the degree of differentiation between goods, because it determines the extent to which disclosure affects firm pricing and relative demand across markets. Regarding optimal disclosure for firms, with substitutes, firm choices are strategic complements and the informed firm thus benefits from sharing its private information with the uninformed firm, because it increases price correlation. As a result, full disclosure is optimal for the informed firm. Full disclosure also maximizes producer surplus, because the uninformed firm also benefits from price correlation as well as from learning

¹See for example media coverage in Fung (2020), Green (2018), Lardieri (2019).

about the state. This result highlights that an informed firm may have incentives to share information even when it has no information to gain in return, because it can influence the pricing of its competitor. Furthermore, I generalize this result by showing that the informed firm's optimal disclosure doesn't rely on the linearity of demand: when the informed firm's expected equilibrium profit is supermodular in the state and the choice of the uninformed firm, firms' choices are strategic complements and, accordingly, full disclosure is optimal. I also show that no disclosure is optimal when the informed firm's expected equilibrium profit is submodular.²

Regarding optimal disclosure for consumers, I find that a regulator should restrict information disclosure, at least partially.³ However, some information disclosure is not necessarily detrimental to consumers. First, I show that the optimal disclosure is private, such that the informed firm doesn't observe the signal realization of the uninformed firm. Second, I show that optimal disclosure is determined by the degree of differentiation between goods. In particular, partial disclosure is optimal if firms offer sufficiently close substitutes and no disclosure is optimal otherwise. Information disclosure creates a trade-off for consumers. On the one hand, disclosure reduces the uninformed firm's uncertainty about the state, improving the ability of firms to extract surplus from consumers by increasing price correlation. On the other hand, private partial disclosure introduces uncertainty about the information observed by a firm's competitor. This expands the range of prices in each state, since firms price according to the expected price of its competitor and its expected level of demand. Namely, the uninformed firm may observe a signal which conflicts with the realized state, but the informed firm doesn't observe the signal and therefore cannot adjust, creating a coordination failure. Consumers can benefit from this price heterogeneity by choosing from which firm to buy after observing prices. Overall, the regulator trades-off the opportunity to create this coordination failure in prices with allowing firms to better extract surplus from consumers. The net effect depends on the differentiation between goods, because it determines consumers' willingness to substitute between goods and therefore the extent to which disclosure affects relative demand across firms.

²When the informed firm's profits are supermodular in the state and the choice of the uninformed firm, an increase in the uninformed firm's price has a increasing effect on the informed firm's profits as the state increases. Focusing instead on decision problems, Kolotilin and Wolitzky (2020) shows that supermodularity of a sender's objective function with respect to the state and the receiver's action is a sufficient condition for the optimality of full disclosure.

³Luco (2019) presents empirical evidence that full disclosure can be detrimental for consumers in the gasoline market in Chile.

Combining results for firms and consumers, to maximize expected welfare, the regulator trades off the effect of disclosure on consumers and firms. When firms offer sufficiently differentiated goods, no disclosure maximizes expected welfare, since the expected loss of some disclosure for consumers exceeds the expected gain for firms. Conversely, when firms offer sufficiently close substitutes, full disclosure maximizes welfare. For intermediate levels of differentiation, partial disclosure maximizes welfare.

When partial disclosure is optimal, I also fully characterize the consumer and welfare optimal disclosure policies, which share the same qualitative properties. Signals act as equilibrium price recommendations, recommending a price to each firm conditional on the state subject to obedience constraints. First, I show that the regulator recommends at most two prices. Second, I show that one of the prices is only recommended when the state is low, revealing the state to the uninformed firm. The other price is recommended in both states, obfuscating the level of demand. The optimality of partial disclosure contrasts with work in the literature focusing on firm incentives, highlighting that optimal disclosure can be more nuanced when considering implications for consumers and welfare.

My analysis emphasizes the wide scope for intervention by a regulator, based on product differentiation and their objective function. My results are of particular interest given current policy debates on the use of private information by firms who act as both a trading platform and a competitor in the market. As I show, it can be optimal for a regulator to intervene by completely preventing or forcing information disclosure, or by designing disclosure policies to partially inform the uninformed firm. Therefore, it is crucial to consider the strategic environment to understand the consequences of information sharing between firms, and whether it is beneficial for both firms and consumers.

Related literature. This paper contributes to the literature on strategic information sharing in oligopolies with commitment and the literature on information design in games.⁴ Incentives for information sharing about demand among competing firms with symmetric private information and normally distributed linear demand were first studied in Novshek and Sonnenschein (1982), Clarke (1983) as well as Vives (1984), and later generalized in Raith (1996).⁵ In these papers, firms commit to share their private information with an intermediary, which then discloses a common signal to all firms to maximize industry-wide

⁴Papers like Benoit and Dubra (2006) show that agents' ex-ante and ex-post incentives for information sharing can be disaligned, such that commitment is key.

⁵Other papers in this literature include Gal-Or (1985), Li (1985), Kirby (1988) and Vives (1990). Information sharing about costs are studied in papers like Fried (1984), Gal-Or (1986), Sakai (1986) and Shapiro (1986), in which incentives to share information are reversed.

profits. These papers focus on the producer surplus optimal public disclosure and on the regulation of industry-wide information sharing by trading organizations. They show the optimality of full disclosure for firms when they compete by choosing prices and offer imperfect substitutes. Instead, I study the incentives of an individual firm to share information to influence its competitor's behavior in a setting of informational advantage, in which signals are privately observed and the distribution of the uninformed firm's signal is unrestricted.⁶ My results show that it can be optimal for a firm to unilaterally disclose information about demand to a competitor even without receiving any information in return, because disclosure can influence the behavior of competitors and act as a pricing persuasion device.⁷ Further, full disclosure is not only optimal for the informed firm, but also for producer surplus. The intuition for this result relates to Angeletos and Pavan (2007), who study the social value of information with normally distributed signals and find that producer surplus increases with the precision of both public and private signals.

In contrast with this literature, I also analyze the effects of information disclosure on consumers. Vives (1984) and Calzolari and Pavan (2006) show that information disclosure is not necessarily harmful to consumers. Vives (1984) illustrates this by comparing the utility of a representative consumer across full and no disclosure when firms share symmetric normally distributed private information. Calzolari and Pavan (2006) study a sequential setting in which the Stackelberg leader must provide incentives to consumers to reveal their private information to be able to share it with its follower. They focus on the leader's optimal disclosure policy, whereas I focus on the optimal disclosure for consumers. Regarding welfare, Vives (1984) also shows that full disclosure dominates no disclosure if and only if firms offer sufficiently close substitutes, yet I show that restricting to full and no disclosure is with loss of generality since partial disclosure can be consumer and welfare optimal. My results regarding welfare relate to Ui and Yoshizawa (2015), who study the social value of information restricted to symmetric normally distributed signals and symmetric equilibria. They show that welfare decreases in the precision of private information and increases in the precision of public information if goods are close substitutes, intuitively related to the optimality of either full or private partial disclosure as I fully characterize in this paper.

⁶Bergemann and Morris (2013), Bergemann et al. (2015b) and Eliaz and Forges (2015) analyze producer optimal disclosure in Cournot settings with perfect substitutes and information symmetry. They show that it is with loss of generality to restrict attention to a common and, hence, perfectly correlated disclosure.

⁷In sequential settings, the role of current choices as a costly persuasion device to influence the precision of future information has been studied in Mailath (1989), Mirman et al. (1993), Mirman et al. (1994), Harrington (1995), Keller and Rady (2003), Taylor (2004), Bernhardt and Taub (2015), Bonatti et al. (2017).

More broadly, the paper contributes to the literature on information design in games as studied in papers like Taneva (2019) and Mathevet et al. (2020). I characterize the optimal recommendation mechanism in a Bertrand setting with product differentiation and information asymmetry. In monopoly settings, Bergemann et al. (2015a), Roesler and Szentes (2017) and Ichihashi (2020) analyze the effect of information about buyers' valuation on monopoly pricing and welfare allocation. In a duopoly setting, Armstrong and Zhou (2019) studies competition between firms when consumers observe a private signal about their valuation. In contrast, I focus on the welfare consequences of information disclosure between firms and characterize the optimal information structure for consumers and welfare.

The remainder of the paper is organized as follows: Section 1 presents an example, Section 2 outlines the model and preliminary results, Section 3 derives the informed firm optimal disclosure, Section 4 derives the consumer optimal disclosure, Section 5 derives the producer and welfare optimal disclosures, Section 6 discusses extensions, and Section 7 concludes.

1 Motivating example

Two firms compete by simultaneously setting prices and offer differentiated substitutes. Firm profits depend on the realization of a binary state, $\theta \in \Theta$, which represents the level of demand. For this example, assume that $\Theta = \{1, 2.8\}$, where the low state occurs with probability $\mu_L = \frac{3}{4}$ and the high state with probability $\mu_H = \frac{1}{4}$. Firm i 's demand is given by

$$q_i((p_i, p_{-i}); \theta) = \max\{0, \theta - 2.1p_i + 2p_{-i}\}.$$

The effect of i 's pricing decision on its quantity demanded exceeds the effect of j 's pricing decision, representing the differentiation between goods. Firms' costs are zero.

Firm 1 (the informed firm) observes the state, whereas firm 2 (the uninformed firm) initially has no information beyond the prior. Before the realization of the state, the informed firm commits to an information structure which discloses none, some, or all of its private information to the uninformed firm. For simplicity, in this example, I restrict attention to information structures with binary support: conditional on the state, the uninformed firm privately observes either a low signal s_L or a high signal s_H . In the low state $\theta_L = 1$, the low signal is observed with probability $x_L \in [0, 1]$, whereas the high signal is observed with the complementary probability $1 - x_L$. In the high state $\theta_H = 2.8$, the high signal is observed with probability $x_H \in [0, 1]$ and the low signal with probability $1 - x_H$. Signal s_L is more likely to occur than s_H when the state is low than when state is high, implying that $x_L + x_H \geq 1$. Denote the information structure by (x_L, x_H) , where full (no) disclosure is captured by $x_L = x_H = 1$ ($x_L = x_H = \frac{1}{2}$).

Given the information structure, firms choose a pricing strategy, which consists of a price conditional on their private information to maximize expected profits. Denote by $p_1^*(\theta; (x_L, x_H))$ the informed firm's equilibrium price when the state is θ and by $p_2^*(s; (x_L, x_H))$ the uninformed firm's equilibrium price after observing signal s .

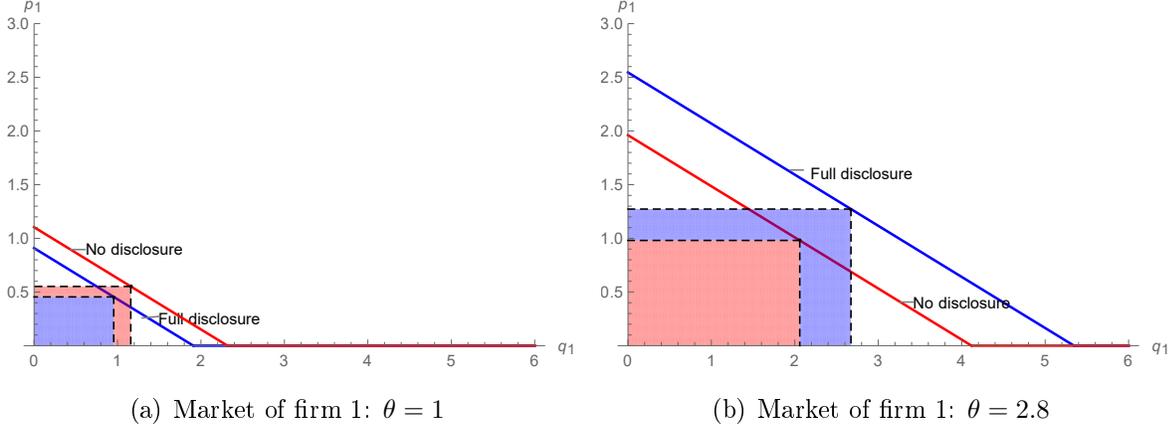


Figure 1: Informed firm's expected demand and profits in each state with full disclosure (blue line and blue shaded region) vs no disclosure (red line and red shaded region).

The informed firm's expected equilibrium profits are strictly increasing in the precision of the information structure.⁸ Hence, it is optimal for it to commit to share all of its private information with the uninformed firm. In Figure 1, I illustrate the optimality of full disclosure, showing its effect on prices, demand and profits in the informed firm's market. Changes in the information observed by the uninformed firm impact equilibrium pricing and, as a result, the expected demand faced by the informed firm. From the informed firm's perspective, a more precise signal increases the uninformed firm's expected equilibrium price in the high state and lowers it in the low state.⁹ As a result, with substitutes, a more precise information disclosure increases (decreases) the informed firm's expected demand in

⁸The informed firm's expected equilibrium profits, $E[\Pi_1^*(x_L, x_H)] = 2.1E[p_1^*(\theta; (x_L, x_H))^2]$, are strictly increasing in both x_L and x_H , because

$$\frac{\partial E[\Pi_1^*(x_L, x_H)]}{\partial x_k} = 2.1 \sum_{\theta \in \Theta} \mu_\theta p_1^*(\theta; (x_L, x_H)) \frac{\partial p_1^*(\theta; (x_L, x_H))}{\partial x_k}, \quad \mu_H \frac{\partial p_1^*(\theta; (x_L, x_H))}{\partial x_k} = -\mu_L \frac{\partial p_1^*(\theta; (x_L, x_H))}{\partial x_k} > 0$$

and the informed firm sets a higher price in the high state.

⁹The uninformed firm's optimal price is linear in its expectation of the state. Then, changes in its pricing across states due to disclosure depends on its prior. After observing the low signal s_L , it lowers its price by a relatively small amount if it already believed that the low state was more likely. In contrast, after observing the high signal s_H , its price increase is bigger, reflecting its change in beliefs about the state.

the high (low) state, shifting the demand outward (inward) in Figure 1. In the high state, the informed firm increases its profits by raising the price on inframarginal consumers who were already buying its product and gaining marginal consumers from the uninformed firm's market. In the low state, the informed firm's profits are lower since it charges a lower price and faces a lower level of demand. Nevertheless, the bigger size of the market implies that the stakes are higher in the high state, such that the gains from a price bonus there outweigh the losses from a price penalty in the low state. As a result, it is optimal for the informed firm to disclose its private information.

However, full disclosure is detrimental for consumers. Consumers are better off with no disclosure between firms, as illustrated in Table 1. Still, while completely restricting disclosure may seem optimal, private partial disclosure can benefit consumers. For instance, private partial disclosure characterized by $(x_L, x_H) = (\frac{1}{2}, 1)$ yields a higher expected consumer surplus than no disclosure, as also illustrated in Table 1.¹⁰

	No: $(x_L, x_H) = (\frac{1}{2}, \frac{1}{2})$	Partial: $(x_L, x_H) = (\frac{1}{2}, 1)$	Full: $(x_L, x_H) = (1, 1)$
p_1^*	$p_1^*(\theta_L; \cdot) = 0.55, p_1^*(\theta_H; \cdot) = 0.98$	$p_1^*(\theta_L; \cdot) = 0.54, p_1^*(\theta_H; \cdot) = 1$	$p_1^*(\theta_L; \cdot) = 0.45, p_1^*(\theta_H; \cdot) = 1.27$
p_2^*	$p_2^*(s_L; \cdot) = p_2^*(s_H; \cdot) = 0.66$	$p_2^*(s_L; \cdot) = 0.49, p_2^*(s_H; \cdot) = 0.76$	$p_2^*(s_L; \cdot) = 0.45, p_2^*(s_H; \cdot) = 1.27$
CS	1.26	1.27	1.18

Table 1: Prices and Consumer surplus comparison with no, partial and full disclosure.

Intuitively, private partial disclosure impacts consumer surplus in two ways. First, it gives the uninformed firm information about the state, improving the ability of firms to extract surplus from consumers. Second, it introduces uncertainty in the pricing of firms, because each firm is uncertain about the information observed by its competitor. This increases the range of prices in each state, such that consumers can benefit from price arbitrage by choosing from which firm to buy after observing prices. In fact, it creates a pricing coordination failure between firms with positive probability, given that they choose prices without observing its competitor's information.

Figure 2 illustrates ex-post consumer surplus with no disclosure and partial disclosure $(x_L, x_H) = (\frac{1}{2}, 1)$ in each market. Each panel illustrates consumer surplus when the realized state is θ and the uninformed firm observes signal s_2 . With this partial disclosure, the uninformed firm observes signal s_H with probability 1 when the state is high, so only three panels per market are displayed.

¹⁰Consumers benefit from partial disclosure when it is private, in which case the informed firm doesn't observe the signal realization of the uninformed firm. If disclosure is public, no disclosure is optimal.

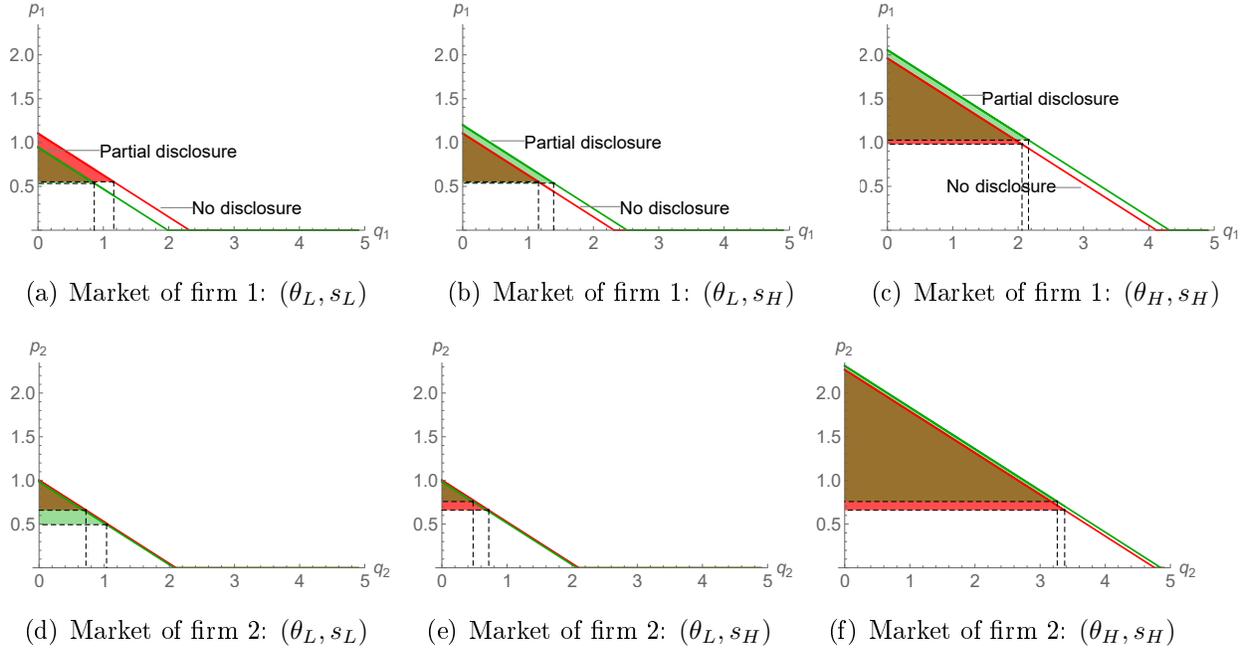


Figure 2: Ex-post consumer surplus with no disclosure (shaded red area) vs partial disclosure $(x_L, x_H) = (\frac{1}{2}, 1)$ (shaded green area).

In the uninformed firm's market, expected consumer surplus decreases with partial disclosure due to changes in its pricing. There is little change in its demand, since the informed firm knows the state and therefore does not substantially change its pricing based on the disclosure policy. As shown in panel (d) (panels (e) and (f)), when the uninformed firm observes the low (high) signal, it decreases (increases) its price, increasing (decreasing) quantity sold and consumer surplus. The net effect on expected consumer surplus is negative due to the larger size of the market in the high state, as illustrated in panel (b) of Figure 3.

In the informed firm's market, expected consumer surplus increases with partial disclosure. In contrast to market 2, the pricing of the informed firm changes little across disclosure policies, such that changes in consumer surplus are driven by changes in the level of demand from the uninformed firm's pricing. In particular, the informed firm's demand shifts inward when the uninformed firm observes the low signal (panel (a)) and outward when it observes the high signal (panels (b) and (c)). Accordingly, most of the gain in consumer surplus occurs when the uninformed firm observes the high signal, especially when the state is low. In that case, there is a mismatch between the signal realization and the state, implying that the uninformed firm charges a high price. However, since the informed firm doesn't observe

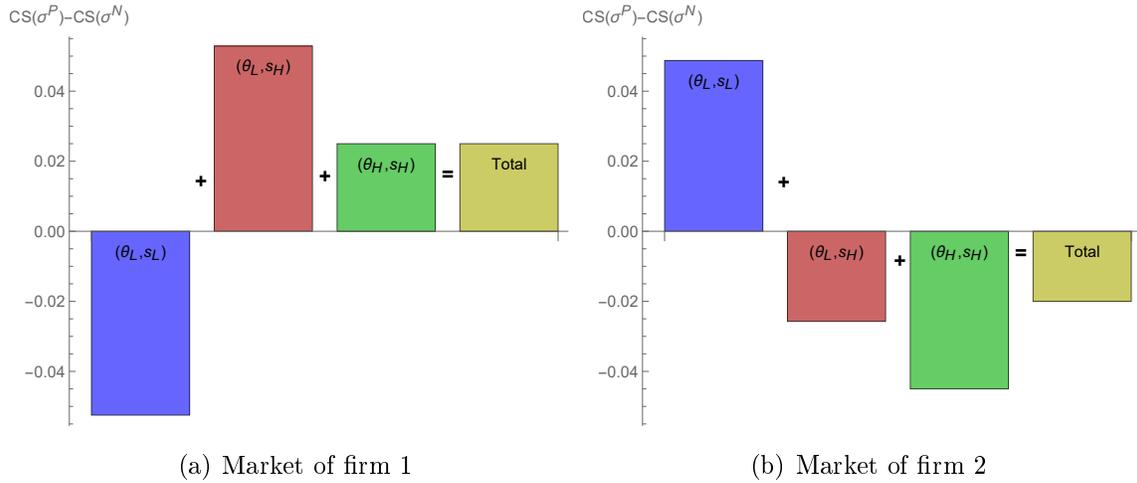


Figure 3: Weighted change in consumer surplus from partial disclosure vs. no disclosure

the uninformed firm's signal, it can't take advantage of the uninformed firm's high price. Through price arbitrage, consumers avoid the high price in the uninformed firm's market by buying from the informed firm at the lower price. Overall, as shown in panel (a) of Figure 3, the expected change in consumer surplus for market 1 is positive and exceeds the loss in market 2, showing that partial disclosure is optimal for consumers.

2 The model

Two symmetric firms offer horizontally differentiated substitutes and compete by simultaneously setting prices. Firm profits depend on the realization of a binary payoff-relevant state, $\theta \in \Theta = \{\theta_L, \theta_H\}$ with $\theta_H > \theta_L > 0$. Firms share a common prior about the state where the probability of $\theta \in \Theta$ is denoted by $\mu_\theta \in (0, 1)$. Firm i 's demand, $q_i((p_i, p_{-i}); \theta)$, is given by

$$q_i((p_i, p_{-i}); \theta) = \max\{0, \theta - ap_i + bp_{-i}\} \quad (1)$$

where a and b are known parameters with $a > b > 0$.¹¹ As can be seen from (1), the state represents the level of demand and, since firms offer substitutes, both the state and the price of the competitor are positive demand shifters which increase quantity demanded at every price. Define δ as the ratio of b and a , which measures the degree of differentiation. As δ converges to 0, goods are more differentiated and as δ converges to 1, goods are closer

¹¹The relationship between a and b implies that demand is more sensitive to a firm's own price than the price of its competitor, ensuring that equilibrium prices are finite.

substitutes. Also, assume that

$$\theta_H < \frac{4a^2 - b^2}{2a^2 - b^2} \theta_L,$$

ensuring that equilibrium prices and quantities are non-negative. Firms' costs are zero.¹² Hence, firm i 's ex-post profits, $\Pi_i : \mathbb{R}_+^2 \times \Theta \rightarrow \mathbb{R}$, correspond to

$$\Pi_i((p_i, p_{-i}); \theta) = p_i \cdot q_i((p_i, p_{-i}); \theta).$$

Given the state and prices (p_i, p_{-i}) , ex-post consumer surplus in the market of firm i is the difference between consumers' ex-post willingness to pay for the good and the equilibrium price. The ex-post willingness to pay of consumers in the market of firm i is characterized by the ex-post inverse demand, $p_i(q_i; p_{-i}, \theta)$, given by

$$p_i(q_i; p_{-i}, \theta) = \max \left\{ 0, \frac{\theta + bp_{-i} - q_i}{a} \right\}$$

where demand is generated by a continuum of heterogeneous consumers making discrete choices (Armstrong and Vickers, 2015).¹³ Then, ex-post consumer surplus in market of firm i , $CS_i : \mathbb{R}_+^2 \times \Theta \rightarrow \mathbb{R}$, corresponds to

$$CS_i((p_i, p_{-i}); \theta) = \frac{1}{2} \left[\frac{\theta + bp_{-i}}{a} - \frac{\theta + bp_{-i} - q_i((p_i, p_{-i}); \theta)}{a} \right] q_i((p_i, p_{-i}); \theta) = \frac{1}{2a} q_i((p_i, p_{-i}); \theta)^2, \quad (2)$$

where the term in square brackets corresponds to the difference between the demand intercept and the equilibrium price.

¹²Including linear or quadratic costs has no impact on the results.

¹³Assume a continuum of consumers with heterogeneous preferences. Consumer ℓ has valuation $v_{\ell,i}$ for one unit of the good offered by firm i , where $v = (v_{\ell,1}, v_{\ell,2})$ is drawn from a joint cumulative distribution $G(v)$. Consumer ℓ attaches no value to more than one unit of either good and wishes to buy either a single unit of one good or to not buy any of them. Then, consumer ℓ buys from firm i if $v_{\ell,i} - p_i \geq \max_{j \neq i} \{0, v_{\ell,j} - p_j\}$, where the outside option is normalized to zero. The demand for product i , $q_i(p)$, is then the measure of consumers ℓ who satisfy $v_{\ell,i} - p_i \geq \max_{j \neq i} \{0, v_{\ell,j} - p_j\}$. Armstrong and Vickers (2015) shows that the linear demand model defined by (1) can be micro-founded by this discrete choice model, since

$$\frac{\partial q_i((p_i, p_{-i}); \theta)}{\partial p_{-i}} = \frac{\partial q_{-i}((p_i, p_{-i}); \theta)}{\partial p_i} \text{ for all } i \text{ and } \frac{\partial^2 \sum_{i \in \{1,2\}} q_i((p_i, p_{-i}); \theta)}{\partial p_i \partial p_{-i}} \leq 0.$$

In this context, consumer ℓ who buys from firm i receives surplus $v_{\ell,i} - p_i$ and the consumer surplus in market i is simple the sum of the surpluses of consumers ℓ who purchase good i . Given that there is a continuum of consumers, this coincides with (2). Furthermore, since consumers buy at most one product, total ex-post consumer surplus is simply the sum across markets. See Choné and Linnemer (2020) for a survey of micro-foundations of a linear demand system.

Information environment. Firm 1 (the informed firm) knows the state, whereas firm 2 (the uninformed firm) initially has no information beyond the common prior. Assume that a designer can restrict (or require) information sharing between firms by choosing the information observed by the uninformed firm. The designer selects and commits to an information structure before the realization of the state, which discloses none, some, or all of the informed firm’s private information to the uninformed firm. Let S_2 be the set of signal realizations observed by firm 2. Signal realizations are private.¹⁴ An information structure consists of a set of signal realizations S_2 and a family of conditional distributions $\psi_2 : \Theta \rightarrow \Delta(S_2)$.

Pricing game. Given the information structure (S_2, ψ_2) , firms play a pricing game in which they condition their pricing choices on their information by selecting mappings

$$\hat{\beta}_1 : \Theta \rightarrow \Delta(\mathbb{R}_+) \text{ and } \hat{\beta}_2 : S_2 \rightarrow \Delta(\mathbb{R}_+)$$

to maximize their expected profits. Specifically, the timing is as follows: (i) the designer selects and commits to an information structure (S_2, ψ_2) ; (ii) the state θ is realized; (iii) the signal is realized and privately observed by the uninformed firm according to (S_2, ψ_2) ; (iv) firms update their beliefs according to Bayes’ rule and simultaneously choose prices; (v) payoffs are realized.

The solution concept is Bayes Nash equilibrium (BNE). A strategy profile $(\hat{\beta}_1, \hat{\beta}_2)$ is a BNE if, for all $p_i \in \mathbb{R}_+$ selected with positive probability,

$$\int_{S_2} \int_{\mathbb{R}_+} \Pi_1((p_1, p_2); \theta) d\hat{\beta}_2(p_2|s_2) d\psi_2(s_2|\theta) \geq \int_{S_2} \int_{\mathbb{R}_+} \Pi_1((p'_1, p_2); \theta) d\hat{\beta}_2(p_2|s_2) d\psi_2(s_2|\theta) \quad (3)$$

for all $p'_1 \in \mathbb{R}_+$ and $\theta \in \Theta$ and

$$\sum_{\theta \in \Theta} \mu_\theta \int_{\mathbb{R}_+} \Pi_2((p_2, p_1); \theta) d\hat{\beta}_1(p_1|\theta) \geq \sum_{\theta \in \Theta} \mu_\theta \int_{\mathbb{R}_+} \Pi_2((p'_2, p_1); \theta) d\hat{\beta}_1(p_1|\theta) \quad (4)$$

for all $p'_2 \in \mathbb{R}_+$ and $s_2 \in S_2$. Denote by $\hat{\mathcal{E}}(S_2, \psi_2)$ the set of BNE in the pricing game.

For any information structure (S_2, ψ_2) , the existence and uniqueness of the BNE is guaranteed by Ui (2016), which provides sufficient conditions for the existence and uniqueness of the BNE in Bayesian games with concave and continuously differentiable payoff functions. This result is formalized in Lemma 1. The proofs of this result and all subsequent others are in Appendix A.2.

¹⁴Allowing for public signals has no effect on the optimal disclosure for firms. However, consumers are better off when signals are private for any disclosure policy. Then, a designer concerned about consumers would optimally commit to private disclosure. See Lemma 8.

Lemma 1 For all information structures (S_2, ψ_2) , the set of BNE in the pricing game $\hat{\mathcal{E}}(S_2, \psi_2)$ is a singleton.

Information disclosure. The choice of information structure (S_2, ψ_2) determines the equilibrium in the pricing game. The designer chooses an information structure to maximize its ex-ante expected payoff such that $(\hat{\beta}_1^*(p_1|\theta), \hat{\beta}_2^*(p_2|s_2))$ is the BNE of the pricing game. I consider four objective functions for the designer:

i) Informed firm expected profits:

$$\mathbb{E}[\Pi_1((p_1, p_2); \theta)] = \sum_{\theta \in \Theta} \mu_\theta \int_{S_2} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \Pi_1((p_1, p_2); \theta) d\hat{\beta}_2^*(p_2|s_2) d\hat{\beta}_1^*(p_1|\theta) d\psi_2(s_2|\theta)$$

ii) Expected Consumer surplus:

$$\mathbb{E}[CS((p_1, p_2); \theta)] = \frac{1}{2a} \sum_{i \in \{1,2\}} \sum_{\theta \in \Theta} \mu_\theta \int_{S_2} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} q_i((p_i, p_{-i}); \theta)^2 d\hat{\beta}_1^*(p_1|\theta) d\hat{\beta}_2^*(p_2|s_2) d\psi_2(s_2|\theta)$$

iii) Expected Producer surplus:

$$\sum_{i=1,2} \mathbb{E}[\Pi_i((p_1, p_2); \theta)] = \sum_{i \in \{1,2\}} \sum_{\theta \in \Theta} \mu_\theta \int_{S_2} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \Pi_i((p_i, p_{-i}); \theta) d\hat{\beta}_1^*(p_1|\theta) d\hat{\beta}_2^*(p_2|s_2) d\psi_2(s_2|\theta)$$

iv) Expected Welfare:

$$\mathbb{E}[W((p_1, p_2); \theta)] := \mathbb{E}[CS((p_1, p_2); \theta)] + \sum_{i=1,2} \mathbb{E}[\Pi_i((p_1, p_2); \theta)].$$

The interpretation of the role of the designer varies depending on their objective function. If the designer's objective is to maximize the informed firm's expected profits, then it is as if the informed firm is choosing how much information to disclose to its competitor. If the designer's objective is to maximize expected producer surplus, it is as if there is a collusive agreement between firms to determine optimal disclosure of information among them. Lastly, if the designer's objective is to maximize expected consumer surplus or welfare, the interpretation of the designer is as a regulator.

The main effects of information disclosure on welfare are captured by the trade offs arising from the informed firm and the consumer optimal disclosures. The insights obtained by analyzing them extend to the producer surplus and welfare optimal disclosures.

2.1 Preliminary results

2.1.1 Equivalence to recommendation mechanisms

This section simplifies the information design problem by constraining the set of information structures. Taneva (2019) shows that it is without loss of generality to restrict attention to information structures where signals are equilibrium recommendations conditional on the state. I present an extension to compact action spaces and bounded, continuous real-valued payoff functions, restricting attention to $p_i \in [0, \frac{\theta_H}{a-b}]$ for all $i \in \{1, 2\}$.¹⁵ In a recommendation mechanism, the pricing rule $\sigma : \Theta \rightarrow \Delta\left([0, \frac{\theta_H}{a-b}]^2\right)$ recommends a price for each firm such that the obedience constraints are satisfied, ensuring that firms are willing to follow the price recommendation. Any pricing rule which satisfies the obedience constraints is a Bayes Correlated Equilibrium (BCE), as introduced by Bergemann and Morris (2013). A pricing rule $\sigma : \Theta \rightarrow \Delta\left([0, \frac{\theta_H}{a-b}]^2\right)$ is a BCE if

$$\sum_{\theta \in \Theta} \mu_\theta \int_{p_{-i} \in [0, \frac{\theta_H}{a-b}]} \Pi_i((p_i, p_{-i}), \theta) d\sigma((p_i, p_{-i})|\theta) \geq \sum_{\theta \in \Theta} \mu_\theta \int_{p_{-i} \in [0, \frac{\theta_H}{a-b}]} \Pi_i((p'_i, p_{-i}), \theta) d\sigma((p_i, p_{-i})|\theta) \quad (5)$$

for all $p_i \in \text{supp } \sigma$, $p'_i \in [0, \frac{\theta_H}{a-b}]$, and $i \in \{1, 2\}$.

Consider an analogous information environment in which both firms observe a private signal about the state such that the informed firm's signal is perfectly informative. In what follows, I show that it is without loss of generality to interpret signals (s_1, s_2) as equilibrium recommendations in which each signal recommends a price to each firm. Define the information structure as the joint distribution of signals. Let S_i be the set of private signal realizations for firm i . An information structure consists of a set of signal realizations and a family of conditional distributions $\psi : \Theta \rightarrow \Delta(S)$, where $S = S_1 \times S_2 = \{s_L, s_H\} \times S_2$. Let $\psi_i : \Theta \rightarrow \Delta(S_i)$ be the marginal distribution of signal $s_i \in S_i$ given the information structure (S, π) . The marginal distribution ψ_1 is fully informative about the state, which implies that the probability of observing signal s_k conditional on state θ_k is 1.

Given the information structure (S, ψ) , firms play a pricing game in which they condition their pricing choices on their signal realization by selecting a mapping $\beta_i : S_i \rightarrow \Delta([0, \frac{\theta_H}{a-b}])$ to maximize their expected profits. A strategy profile (β_1, β_2) is a BNE if, for all $p_i \in [0, \frac{\theta_H}{a-b}]$

¹⁵This restriction is without loss of generality, since any price above $\frac{\theta_H}{a-b}$ induces no trade and profits of zero for firm i .

with $\beta_i(p_i|s_i) > 0$ for all i , we have

$$\begin{aligned} & \sum_{\theta \in \Theta} \mu_\theta \int_{S_{-i}} \int_0^{\frac{\theta_H}{a-b}} \Pi_i((p_i, p_{-i}); \theta) d\beta_{-i}(p_{-i}|s_{-i}) d\psi((s_i, s_{-i})|\theta) \\ & \geq \sum_{\theta \in \Theta} \mu_\theta \int_{S_{-i}} \int_0^{\frac{\theta_H}{a-b}} \Pi_i((p'_i, p_{-i}); \theta) d\beta_{-i}(p_{-i}|s_{-i}) d\psi((s_i, s_{-i})|\theta) \end{aligned} \quad (6)$$

for all $p'_i \in [0, \frac{\theta_H}{a-b}]$, $s \in S$ and $i \in \{1, 2\}$. Denote by $\mathcal{E}(S, \psi)$ the set of BNE in the pricing game.

First, Lemma 2 is an equivalence result stating that every possible BCE distribution can be replicated as a BNE by appropriately choosing the information structure. Intuitively, any correlation between obedient pricing choices can be generated as a BCE. In a BNE, all the correlation between pricing choices is generated through the information structure (S, ψ) .

Lemma 2 (Taneva, 2019) *The set of BCE coincides with $\cup_{(S, \psi)} \mathcal{E}(S, \psi)$.*

Second, Lemma 3 implies that it is without loss of generality to restrict attention to recommendation mechanisms. Formally, an information structure (S, ψ) is a recommendation mechanism if $S = [0, \frac{\theta_H}{a-b}]^2$. In a recommendation mechanism, signals act as pricing recommendations which firms are willing to follow as long as their competitor does as well.

Lemma 3 (Taneva, 2019) *For every $\sigma \in \cup_{(S, \psi)} \mathcal{E}(S, \psi)$, there exists a recommendation mechanism $([0, \frac{\theta_H}{a-b}]^2, \sigma)$ such that $\sigma \in \mathcal{E}([0, \frac{\theta_H}{a-b}]^2, \sigma)$.*

2.1.2 Existence of optimal recommendation mechanism

The existence of the optimal recommendation mechanism stated in Lemma 4 is guaranteed by the Weierstrass extreme value theorem. First, the existence of correlated equilibria for games in which players receive private signals and simultaneously choose actions from compact sets is established in Stinchcombe (2011). Second, the set of BCE is compact in the weak* topology, since it is the set of all probability measures on a compact set.¹⁶ Then, the designer's problem is to maximize a continuous function of σ over a non-empty compact set.

Lemma 4 *The optimal recommendation mechanism exists.*

¹⁶With full disclosure, equilibrium prices are

$$p^F(\theta) = \frac{\theta}{2a - b}.$$

It follows that firms have no incentives to set prices above $p^F(\theta_H)$ or below $p^F(\theta_L)$, because such prices would never be part of a BNE of the pricing game. Hence, the support of any obedient recommendation mechanism must be a subset of $[p^F(\theta_L), p^F(\theta_H)]^2$. See Appendix A.2 for a formal argument.

3 Informed firm optimal disclosure

In this section, I consider the case in which the informed firm directly determines its optimal information disclosure. That is, assume that the designer's objective is to maximize the informed firm's expected profits. The informed firm chooses a feasible obedient recommendation mechanism σ to maximize its expected equilibrium profits in the pricing game. Proposition 1 states that it is optimal for the informed firm to share its information.

Proposition 1 (Informed firm optimal disclosure) *It is optimal for the informed firm to fully reveal its private information to the uninformed firm.*

The optimal disclosure policy is determined by the fact that pricing choices are strategic complements, which determines the effect of changes in the precision of the uninformed firm's signal on the informed firm's expected profits. In particular, the informed firm's expected equilibrium profits, $\mathbb{E}_{(\mu,\sigma)}[\Pi_1^*((p_1, p_2); \theta)]$, can be expressed as

$$\mathbb{E}_{(\mu,\sigma)}[\Pi_1^*((p_1, p_2); \theta)] = a\mathbb{E}_\mu \left[\left(\frac{\theta + b\mathbb{E}_\sigma[p_2|\theta]}{2a} \right)^2 \right].$$

Then, maximizing the informed firm's expected equilibrium profits is equivalent to maximizing the distance between the expected equilibrium prices set by the uninformed firm across states, $\mathbb{E}_\sigma[p_2|\theta_L]$ and $\mathbb{E}_\sigma[p_2|\theta_H]$, because of the convexity of the informed firm's expected equilibrium profits with respect to the conditional expectation of the uninformed firm's price. With no disclosure, the uninformed firm sets one price in both states. Increasing the precision of the signal observed by the uninformed firm increases the correlation between its expected price and the state and, therefore, variation in its expected price. As a result, full disclosure maximizes the informed firm's expected profits.

Intuitively, increasing the precision of the signal observed by the uninformed firm increases its certainty about the state, increasing (decreasing) expected demand when its posterior beliefs suggest that the high (low) state is more likely. Accordingly, the uninformed firm increases its expected equilibrium price in the high state and decreases it in the low state. The higher competitor price translates to a higher demand for the informed firm, allowing it to increase its own price in the high state, since firms offer substitutes. The informed firm then sells a higher quantity at a higher price, increasing profits. The opposite is true in the low state since it charges a lower price and faces lower demand, but the expected profit gain in the high state exceeds the expected loss in the low state given the larger size of the market in the high state. Hence, the informed firm benefits from price correlation

and its expected equilibrium profits increase in the precision of the uninformed firm's signal. Since this precision is maximized by full disclosure, it is optimal for the informed firm to fully disclose its private information.

The optimality of full disclosure doesn't rely on the linearity of demand. As formalized in Proposition 6 in Appendix A.4, full disclosure is optimal if the informed firm's expected equilibrium profits are supermodular in the state and the uninformed firm's price. I also show that no disclosure is optimal if the informed firm's expected equilibrium profits are submodular in the state and the uninformed firm's price. This implies that it is optimal for the informed firm to reveal no information to its competitor when they compete by setting prices and offer differentiated complement goods. Kolotilin and Wolitzky (2020) obtain a related result in a setting in which the sender and the receiver do not interact. They show that supermodularity of the sender's objective function with respect to the state and the receiver's action is a sufficient condition for the optimality of full disclosure in decision problems. My results strengthen findings from previous work (Vives (1984), Vives (1990) and Raith (1996)), showing the optimality of either full or no information disclosure in a setting of information asymmetry where the distribution of the uninformed firm's signal and the correlation with the informed firm's signal are unrestricted.¹⁷ One takeaway from my results is that it can be optimal for a firm to disclose information to a competitor even when it has no information to gain in return, because the firm can use disclosure to influence competitor prices.

4 Consumer optimal disclosure

In this section, I interpret the designer as a regulator whose objective is to determine the consumer optimal disclosure. In particular, assume that the designer's objective is to choose an obedient price recommendation mechanism σ that maximizes expected consumer surplus.

The optimal disclosure is formalized in Proposition 2. Partial disclosure is optimal for consumers when firms offer sufficiently close substitutes. Otherwise, no disclosure is optimal. Disclosure allows the uninformed firm to better tailor its price to the state, which in turn increases the ability of firms to extract surplus from consumers. But, private disclosure allows consumers to arbitrage prices by creating a potential coordination failure in firm pricing.

¹⁷They also strengthen results from Novshek and Sonnenschein (1982), Clarke (1983) and Gal-Or (1985), given that Cournot with substitutes (complements) is equivalent to Bertrand with complements (substitutes) from the point of view of firms, as discussed in Raith (1996).

Proposition 2 (Consumer optimal disclosure) *If the designer's objective is to maximize expected consumer surplus, there exists an $\hat{\alpha} \in (0, \hat{c}]$ such that partial disclosure is optimal if $\delta \in (\hat{\alpha}, 1)$ and no disclosure is optimal otherwise, where \hat{c} is the cutoff for binary information structures.*

Intuitively, the impact of disclosure on consumer surplus is determined through two channels. On the one hand, disclosure provides the uninformed firm with information about the state, which increases the correlation between its pricing and the state. Indirectly, this also increases pricing correlation across firms, since the informed firm knows the state. Accordingly, in expectation, firms more accurately tailor their prices to the demand they face, allowing firms to better extract surplus from consumers. On the other hand, it creates uncertainty in firms' pricing decisions, because both firms now have private information. Even if disclosure increases expected price correlation between firms, uncertainty about the signal realization observed by their competitor generates a pricing coordination failure with positive probability. That is, the uninformed firm may observe a signal realization that mismatches with the state, setting a price tailored to the incorrect state. In contrast, the informed firm sets a price tailored to the realized state. When the mismatch occurs and firms set different prices, consumers benefit from arbitraging prices by selecting from which firm to purchase after observing prices. Private disclosure can thus benefit consumers.

The relative impact of these effects on consumer surplus is determined by the degree of differentiation between goods. When goods are close substitutes, a price differential between firms caused by partial private disclosure induces a large segment of the market to buy from the firm with a comparatively low price, creating large gains in consumer surplus with positive probability. In contrast, when goods are not close substitutes, the pricing coordination failure has little impact on the demand that firms face, yielding negligible benefits from price arbitrage. Accordingly, when goods are sufficiently close substitutes, private partial disclosure creates a large enough expected benefit from a potential price coordination failure to provide incentives for the regulator to impose partial disclosure. Otherwise, no disclosure is optimal.

The sketch of the proof is as follows. First, I verify that no disclosure yields a higher expected consumer surplus than full disclosure for all demand parameters and prior distributions of the state. Second, I show that the difference between the expected consumer surplus with partial disclosure σ and no disclosure σ^N , denoted by $\Delta\mathbb{E}[CS](\sigma)$, is a continuous and strictly increasing function of δ for all σ . Third, I show that the optimal disclosure is determined by the degree of substitution δ . As a first step, I restrict attention to binary

information structures and show that there exists a cutoff \hat{c} in the degree of differentiation above which partial disclosure is optimal.¹⁸ As a second step, I show that no disclosure σ^N yields a higher expected consumer surplus than any recommendation mechanism σ when $\delta \rightarrow 0$. Hence, the intermediate value theorem implies that there exists a cutoff in the degree of differentiation, $\hat{a} \in (0, \hat{c}]$, above which partial disclosure is optimal and in particular better than no disclosure.

More specifically, in the unique BNE of the pricing game, the informed firm's optimal pricing strategy is determined by the price recommendation made to the uninformed firm. Define $\sigma(p_2|\theta)$ as the price recommendation to the uninformed firm given the equilibrium price recommendations $\sigma((p_1, p_2); \theta)$. The informed firm's recommended prices satisfy

$$p_1^\sigma(\theta) = \frac{\theta + b \int p_2 d\sigma(p_2|\theta)}{2a}.$$

This implies that, for any obedient recommendation mechanism $\sigma((p_1, p_2); \theta)$, it is sufficient to pin down $\sigma(p_2|\theta)$ since it determines both firms' equilibrium pricing decisions. Then, relying on properties of expectations and variances, the difference between expected consumer surplus with partial disclosure $\mathbb{E}_{(\mu, \sigma)}[CS((p_1, p_2); \theta)]$ and no disclosure $\mathbb{E}_{(\mu, \sigma^N)}[CS((p_1, p_2); \theta)]$ is a linear combination of three moments of $\sigma(p_2|\theta)$ that can be expressed as

$$\Delta\mathbb{E}[CS](\sigma) = C_1(\delta, a)\mathbb{V}_{(\mu, \sigma)}[p_2] - C_2(\delta)Cov_{(\mu, \sigma)}(\theta, p_2) - C_3(\delta, b)\mathbb{V}_\mu[\mathbb{E}_\sigma[p_2|\theta]],$$

where $C_k(\cdot)$ is strictly positive for all $k \in \{1, 2, 3\}$, $\mathbb{V}_{(\mu, \sigma)}[p_2]$ represents the variance of the uninformed firm's price, $Cov_{(\mu, \sigma)}(\theta, p_2)$ represents the covariance between the uninformed firm's price and the state and $\mathbb{V}_\mu[\mathbb{E}_\sigma[p_2|\theta]]$ denotes the variance of the conditional expectation of the uninformed firm's price conditional on the state.

These three moments determine the impact of information disclosure on consumer surplus through relative changes in the level of demand and pricing in each market.

- i) The expected gain in consumer surplus increases in the variance of the uninformed firm's price, $\mathbb{V}_{(\mu, \sigma)}[p_2]$, because it increases the opportunity for consumers to arbitrage price differences between firms and substitute between them.
- ii) The expected gain decreases in the covariance between the uninformed firm's price and the state, $Cov_{(\mu, \sigma)}[p_2, \theta]$, since it captures surplus extraction from consumers through the uninformed firm's better pricing decision.
- iii) The expected gain decreases in the variance of the expectation of p_2 conditional on the state, $\mathbb{V}_\mu[\mathbb{E}_\sigma[p_2|\theta]]$, because it captures the effect of disclosure on the informed

¹⁸Proposition 3 shows that the consumer optimal disclosure in fact has binary support.

firm's pricing. In particular, an increase of $\mathbb{V}_\mu[\mathbb{E}_\sigma[p_2|\theta]]$, reduces the informed firm's uncertainty about the uninformed firm's pricing and increases price correlation between firms.

Therefore, $Cov_{(\mu,\sigma)}[p_2, \theta]$ and $\mathbb{V}_\mu[\mathbb{E}_\sigma[p_2|\theta]]$ measure the loss in expected consumer surplus from disclosure due to the increased ability of firms to extract surplus from consumers. Increasing information disclosure increases all three moments, but their relative magnitude is determined by the degree of differentiation which pins down firms' optimal pricing and how willing consumers are to substitute between goods. In particular, the benefit for consumers increases as firms offer closer substitutes, whereas the ability of firms to extract surplus from consumers decreases since this decreases their market power. As a result, partial disclosure is optimal for consumers when firms offer sufficiently close substitutes.

Next, when partial disclosure is optimal, I characterize the optimal partially informative recommendation mechanism in Proposition 3. The optimal price recommendation mechanism recommends at most two prices: a low price only recommended in the low state and a high price recommended in both states. The recommended prices maximize the uninformed firm's expected profits given its beliefs about the state. Then, the optimal price recommendation mechanism is characterized by the probability of recommending the low price in the low state, denoted by λ^* , where λ^* determines the recommended prices \hat{p}_L and \hat{p}_H and is chosen to maximize expected consumer surplus subject to firm optimal pricing.¹⁹

Proposition 3 (Consumer optimal recommendation mechanism) *Any consumer optimal recommendation mechanism recommends at most two prices. If an optimal mechanism discloses information, then there exists an optimal mechanisms that recommends one price \hat{p}_L only when the state is low and another price \hat{p}_H in both states where*

$$\hat{p}_L = \frac{4a^2[1 - \mu_L\lambda^*]\theta_L + b^2\mu_H[(1 - \lambda^*)\theta_H - \theta_L]}{(2a - b)[4a^2(1 - \mu_L\lambda^*) - b^2\mu_H\lambda^*]},$$

$$\hat{p}_H = \frac{4a^2[\mu_H\theta_H + \mu_L(1 - \lambda^*)\theta_L] - b^2\mu_H\lambda^*\theta_H}{(2a - b)[4a^2(1 - \mu_L\lambda^*) - b^2\mu_H\lambda^*]},$$

and $\lambda^* := \sigma(\hat{p}_L|\theta_L) \in (0, 1)$ is

$$\lambda^* = \frac{4[\delta(1 - 3\delta^2) + 6(1 - \delta^2)]}{\mu_H\delta^5 + 2\mu_H\delta^4 - (12 - \mu_H)\delta^3 - 6(4 - \mu_H)\delta^2 + 4(1 - \mu_H)\delta + 24(1 - \mu_H)}.$$

¹⁹In this context, the price coordination failure occurs when the low state is realized and the uninformed firm is recommended to price high.

Intuitively, consumers gain from disclosure when there are differences in firms' pricing. In particular, when the state is high and the informed firm sets a corresponding high price, it would be good for consumers for the uninformed firm to observe a low price recommendation, because this allows a segment of consumers to substitute goods and purchase from the uninformed firm at a low price. Similarly, it would be best for consumers for the uninformed firm to observe a high price recommendation in the low state, since the informed firm is already setting a low price.

Moreover, in the high state, recommending an intermediate price to the uninformed firm rather than a low price would provide less benefit to consumers, implying that it is optimal for consumers to recommend at most two prices in the that state. Given that no intermediate price would be recommended in the high state, an intermediate price recommendation would reveal to the uninformed firm that the state is low, but the uninformed firm would only be willing to set the low price in that case. Hence, an optimal price recommendation mechanism recommends at most two prices.

Lastly, with linear demand, recommending a unique price in the low state or the high state is equivalent, since both options yield the same expected consumer surplus given that consumers benefit from price differentials induced by uncertainty among firms. This implies that there exists an optimal price recommendation mechanism, characterized in Proposition 3, which recommends at most two prices with a unique price recommended in the high state. This price recommendation mechanism minimizes the level of prices set by firms.

The sketch of the proof of Proposition 3 is as follows. First, fixing an arbitrary θ , I show that, for any partially informative obedient recommendation mechanism σ , at most two prices are recommended in state θ if only one price is recommended when the state is $\theta' \neq \theta$. If a unique price \hat{p} is recommended when the state is θ' , observing any other recommendation $p_2 \neq \hat{p}$ reveals to the uninformed firm that the state is $\theta \neq \theta'$. When the uninformed firm knows that the level of demand is θ , the obedience constraint implies that there is a unique price that it is willing to set. As a result, it is not possible to recommend more than two obedient prices across states. Second, I show that it is optimal for the regulator to recommend a unique price when the state is θ' . These results imply that the optimal information structure sends at most two price recommendations. Further, it is fully characterized by the probability of recommending one of the optimal prices when the state is θ . Since the benefit from partial disclosure is a consequence of the induced uncertainty between firms, the choice of the state θ in which two prices are recommended is inconsequential for consumers because of the linearity of demand. Without loss of generality,

I focus on the case in which two prices are recommended in the low state.²⁰

Proposition 3 also implies that the cutoff for the optimality of partial disclosure coincides with the cutoff for binary structures ($\hat{\alpha} = \hat{c}$). Then, Figure 4 illustrates the optimal disclosure policy. Partial disclosure maximizes expected consumer surplus if the degree of differentiation δ belongs to the blue shaded region, while no disclosure is optimal for consumers if δ belongs to the red shaded region. Depending on the demand parameters and distribution of the state, the optimal partial disclosure can increase expected consumer surplus by up to 2% with respect to no disclosure and 10% with respect to full disclosure.²¹



Figure 4: Consumer optimal disclosure

5 Producer surplus and Welfare optimal disclosure

Information disclosure impacts surplus allocation between firms and between firms and consumers, with potential implications for total welfare. In this section, I first characterize the disclosure policy that maximizes expected producer surplus and, combining this result with the consumer optimal disclosure, I derive the expected welfare maximizing disclosure policy.

²⁰With linear demand and binary price recommendations, characterized by $x_\ell = \mathbb{P}(p_2 = p_\ell | \theta = \theta_\ell)$ with $\ell \in \{L, H\}$, the first order conditions of the regulator's maximization problem are collinear. As a result, it is possible to set either x_L or x_H to 1 since the optimality conditions define a relationship between them. In contrast, preliminary results suggest that with a quadratic demand given by $q_i(p_i, p_{-i}; \theta) = \max\{0, \theta + bp_{-i} - ap_i - cp_i^2\}$ where c is positive and sufficiently small, it is optimal to recommend two prices in the low state and one price in the high state. Similarly, when c is negative and sufficiently small, it is optimal to recommend two prices in the high state and one price in the low state.

²¹Given demand parameters, a and b and the distribution of the state, determined by θ_L , θ_H and μ_H , define $\eta(a, b, \theta_L, \theta_H, \mu_H)$ as the maximum expected consumer surplus given by

$$\eta(a, b, \theta_L, \theta_H, \mu_H) := \max_{\lambda \in [0, 1]} \mathbb{E}[CS((p_1, p_2); \theta)].$$

Then, the maximum increases in consumer surplus compared to no and full disclosure are obtained by maximizing the following functions

$$\max_{(a, b, \theta_L, \theta_H, \mu_H)} \frac{\eta(a, b, \theta_L, \theta_H, \mu_H) - \mathbb{E}_{(\mu, \sigma^N)}[CS((p_1, p_2); \theta)]}{\mathbb{E}_{(\mu, \sigma^N)}[CS((p_1, p_2); \theta)]} \text{ or } \max_{(a, b, \theta_L, \theta_H, \mu_H)} \frac{\eta(a, b, \theta_L, \theta_H, \mu_H) - \mathbb{E}_{(\mu, \sigma^F)}[CS((p_1, p_2); \theta)]}{\mathbb{E}_{(\mu, \sigma^F)}[CS((p_1, p_2); \theta)]},$$

with respect to feasible demand parameters and parameters governing the distribution of the state.

5.1 Producer Surplus optimal disclosure

In this section, I interpret the designer as a collusive agreement between firms whose objective is to choose a disclosure policy to maximize expected producer surplus given by

$$\sum_{i=1,2} \mathbb{E}_{(\mu,\sigma)}[\Pi_i((p_i, p_{-i}); \theta)] = \sum_{i \in \{1,2\}} \sum_{\theta \in \Theta} \mu_\theta \int \Pi_i((p_i, p_{-i}); \theta) d\sigma((p_i, p_{-i})|\theta).$$

First, it is optimal for the uninformed firm to learn the state, as stated in Lemma 5, because it increases the correlation between its pricing decisions and the state.

Lemma 5 *The expected profits of the uninformed firm are maximized by full disclosure.*

Proposition 1 and Lemma 5 indicate that full disclosure is optimal for both firms, because it maximizes both the informed and uninformed firm's expected profits. Thus, full disclosure maximizes expected producer surplus.

5.2 Welfare optimal disclosure

Assume that the designer, interpreted as a regulator, wants to maximize expected welfare, defined as the sum of expected consumer and producer surplus. To maximize expected welfare, the regulator trades off the effect of information disclosure on firms and consumers, given their conflicting preferences over disclosure policies. In particular, firms' expected profits are maximized by full disclosure, whereas expected consumer surplus is maximized by no or partial disclosure. However, the benefits from disclosure for both firms and consumers increase as firms offer closer substitutes. As a result, the optimal disclosure is again determined by the degree of differentiation, as stated in Proposition 4.

Proposition 4 (Welfare optimal disclosure) *If the designer's objective is to maximize expected welfare, there exists $\tilde{\alpha}_1 \in (0, 1)$ and $\tilde{\alpha}_2 \in (0, 1)$ such that $\tilde{\alpha}_1 \leq \tilde{\alpha}_2$ and*

- i) no disclosure is optimal when $\delta \in (0, \tilde{\alpha}_1]$.*
- ii) partial disclosure is optimal when $\delta \in (\tilde{\alpha}_1, \tilde{\alpha}_2)$.*
- iii) full disclosure is optimal when $\delta \in [\tilde{\alpha}_2, 1)$.*

When firms offer sufficiently close substitutes, full disclosure is optimal since it is optimal for firms and their expected gains exceed expected losses for consumers. When firms offer sufficiently differentiated substitutes, no disclosure maximizes expected welfare since it is

optimal for consumers and the expected profit gains for firms from disclosure are small. For intermediate levels of differentiation, partial disclosure is optimal.²²

Ui and Yoshizawa (2015) reach a similar conclusion, restricting attention to symmetric normally distributed private and public signals. When firms offer substitutes, they show that welfare decreases in the precision of private information and increases in the precision of public information, related to the optimality of either partial or full disclosure.

Proposition 5 characterizes the welfare optimal partially informative disclosure policy. Incentives for partial disclosure are driven by the effect of disclosure on consumer surplus. Hence, the qualitative features of the disclosure policy are shared with the consumer optimal one stated in Proposition 3. That is, any optimal partially informative recommendation mechanism has binary support, recommends one price only when the state is low, and another price in both states.

Proposition 5 (Welfare optimal recommendation mechanism) *Any welfare optimal recommendation mechanism recommends at most two prices. If the optimal mechanism discloses information, then it recommends one price \hat{p}_L only when the state is low and another price \hat{p}_H in both states where*

$$\hat{p}_L = \frac{4a^2[1 - \mu_L\lambda]\theta_L + b^2\mu_H[(1 - \lambda)\theta_H - \theta_L]}{(2a - b)[4a^2(1 - \mu_L\lambda) - b^2\mu_H\lambda]},$$

$$\hat{p}_H = \frac{4a^2[\mu_H\theta_H + \mu_L(1 - \lambda)\theta_L] - b^2\mu_H\lambda\theta_H}{(2a - b)[4a^2(1 - \mu_L\lambda) - b^2\mu_H\lambda]},$$

and $\lambda^* := \sigma(\hat{p}_L|\theta_L) \in (0, 1)$ maximizes expected welfare.

The welfare optimal disclosure is illustrated in Figure 4. Full disclosure is optimal if δ is in the yellow shaded region, partial disclosure in the blue shaded region, while no disclosure is optimal in the red shaded region. The optimal partial disclosure can increase welfare by up to 1% with respect no and full disclosure. These results suggest that a regulator whose objective is to maximize welfare faces a trade off between consumer and producer surplus, and must take into account the relationship between markets. It highlights that the task of a regulator can be more nuanced than simply banning or releasing information: the exact design of information matters.

²²Suppose instead that the regulator maximizes the weighted sum of producer and consumer surplus, where $\omega \in [0, 1]$ represents the weight assigned to consumers. The cutoffs $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ increase with ω : for sufficiently high ω , only no or partial disclosure can be optimal; for sufficiently low values of ω , full disclosure is always optimal.

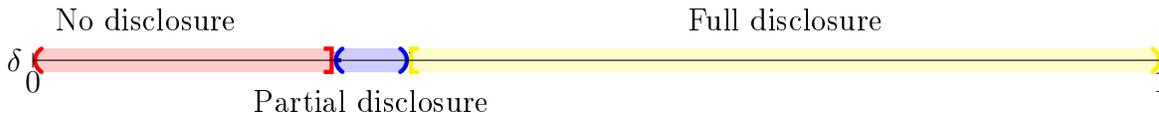


Figure 5: Welfare optimal disclosure

6 Extensions

Firm optimal disclosure with complements. Assume $b \in (-a, 0)$ and define the degree of complementary between goods as $\delta = |\frac{b}{a}|$. When goods are complements, disclosure increases the uninformed firm's profits, as stated in Lemma 5, but reduces the informed firm's profits, as formalized in Proposition 6 in Appendix A.4. The optimal disclosure policy is determined by comparing the gains of the uninformed firm to the losses of the informed firm, which are determined by the degree of complementarity between goods. In particular, if goods are sufficiently complementary ($\delta \geq \hat{\gamma}$), competitor prices have a significant impact on demand. Then, the negative effect of increased pricing correlation on the informed firm's profits exceeds the positive effect of learning about the state on the uninformed firm's profits. As a result, no disclosure is optimal. Otherwise, the informed firm's expected loss from information disclosure is smaller than the uninformed firm's expected gain, implying that full disclosure is optimal. These results are stated in Lemma 6.

Lemma 6 (Producer surplus optimal disclosure) *If the designer's objective is to maximize expected producer surplus, full disclosure is optimal if $\delta \in (0, \hat{\gamma})$. Otherwise, no disclosure is optimal.*

Eliaz and Forges (2015) study the producer surplus optimal disclosure policy in a Cournot duopoly with no private information and unknown demand. They show that it is optimal to fully inform one of the duopolists and disclose no information to the other when firms offer perfect substitutes. They also show that the producer surplus optimal disclosure consists of fully informing both firms when they offer perfect complements. Given the correspondence between Cournot and Bertrand discussed in Raith (1996), my results for complements and substitutes nest theirs, while allowing for more general patterns of complementarity and substitution between goods. Relatedly, Angeletos and Pavan (2007) show that producer surplus increases with the precision of public and private normally distributed signals when firms offer substitutes, related to the optimality of full disclosure. When firms offer complements, they show that producer surplus increases in the precision of private information, but can

decrease in the precision of public information. In my context, this is exemplified by the designer who may have incentives to force information disclosure between firms when they offer complementary goods, decreasing the informational advantage of the informed firm at the benefit of its competitor.

Public signals. Assume that the designer commits to an information structure (S_2, π_2) with public signal realizations. Given the information structure, firms play a pricing game in which they condition their choices on their information by selecting a mapping

$$p_1 : \Theta \times S_2 \rightarrow \Delta(\mathbb{R}_+) \text{ and } p_2 : S_2 \rightarrow \Delta(\mathbb{R}_+)$$

to maximize their expected profits.

Lemma 7 *For any disclosure policy σ , the informed firm's profits are higher with public disclosure than private disclosure.*

Public signals reinforce the informed firm's incentives to disclose information when firms offer substitutes, implying that full disclosure is optimal for the informed firm. When firms offer complements, it is optimal for the informed firm to disclose no information, by the same reasoning as with private signals. In contrast, consumers are better off with private disclosure, since they can benefit from information asymmetry as stated in Lemma 8.

Lemma 8 *For any disclosure policy σ , expected consumer surplus is higher with private disclosure than public disclosure.*

When signals are public, the gain from partial disclosure disappears and, as a result, no disclosure is optimal for consumers as stated in Lemma 9.

Lemma 9 *With public disclosure, no disclosure is optimal for consumers.*

Informed firm as the owner of an online platform. Consider the case in which trade occurs on an online platform run by the informed firm. The informed firm charges the uninformed firm a percentage of its sales for the use of the platform. Given the disclosure policy σ , the informed firm's expected equilibrium payoff is

$$\mathbb{E}_{(\mu, \sigma)}[\Pi_1^*((p_1, p_2); \theta)] = a\mathbb{E}_\mu \left[\left(\frac{\theta + b\mathbb{E}_\sigma[p_2|\theta]}{2a} \right)^2 \right] + \alpha\Pi_2^*(\sigma),$$

where $\alpha \in [0, 1]$ is the percentage of sales charged to the uninformed firm.

The informed firm's incentives for information sharing are minimized by setting $\alpha = 0$, since the uninformed firm always benefits from observing information. When firms offer substitutes, the informed firm optimal disclosure doesn't change with $\alpha > 0$. In this case, the informed firm discloses all of its private information for any $\alpha \in [0, 1]$. When firms offer complements, the informed firm optimal disclosure shares the same qualitative properties as the producer surplus maximizing disclosure with $\alpha = 0$. Full disclosure is optimal if the degree of complementarity is below a certain cutoff, no disclosure is optimal above the cutoff, and the cutoff is an increasing function of α . Furthermore, the producer surplus optimal disclosure remains unchanged, since α represents a transfer between firms. Lastly, the consumer and welfare optimal disclosure also remain unchanged, since they are not affected by transfers between firms.

N symmetric firms with constrained disclosure policies. Consider a setting with $N \geq 3$ firms who compete by choosing prices. The level of demand depends on the state $\theta \in \{\theta_L, \theta_H\}$ with $\theta_H > \theta_L > 0$ such that firms are active in the market in both states. Firms share a common prior about the state, where the probability of θ is denoted by $\mu_\theta \in (0, 1)$. Firm i 's demand is given by

$$q_i(\mathbf{p}) = \theta - ap_i + \frac{b}{N-1} \sum_{j \neq i} p_j$$

where a and b are known parameters with $a > b > 0$. Firms' costs are zero.

The designer commits to an information structure with private signals, denoted by $\hat{\psi}_k$, to share all of the informed firm's private information with k firms and no information with $N - 1 - k$ firms, where $k \in \{0, 1, 2, \dots, N - 1\}$. Firms who observe a perfectly informative signal condition their pricing choices on the state and select a mapping $p^F : \Theta \rightarrow \mathbb{R}_+$ to maximize their expected profits, whereas firms who observe no information select a price $p^N \in \mathbb{R}_+$ to maximize their expected profits. The optimal information disclosure is stated in Lemma 10

Lemma 10 *If the designer's objective is to maximize the informed firm's expected equilibrium profits or to maximize expected producer surplus, it is optimal to share the informed firm's private information with all other firms. In contrast, if the designer's objective is to maximize expected consumer surplus, it is optimal to share the informed firm's private information with $k^*(N, \delta)$ firms where $\frac{k^*(N, \delta)}{N} \leq \frac{2}{3}$.*

First, the informed firm's expected equilibrium profits are maximized by sharing its private information with all other firms because it benefits from price correlation. Similarly, when the designer's objective is to maximize expected producer surplus, it is optimal to share information with all firms, eliminating information asymmetry between firms, allowing them to better extract surplus from consumers.

Second, if the designer's objective is to maximize expected consumer surplus, information disclosure between firms is at least partially restricted. The optimal information structure, characterized by $k^*(N, \delta)$, is determined by the degree of substitution and the number of firms in the market. In particular, it is optimal to not disclose information to any other firm when $\delta \leq \frac{3}{4}$. When $\delta > \frac{3}{4}$, optimal disclosure is determined by the number of firms in the market and δ , as illustrated in Figure 6. In particular, the optimal $k^*(N, \delta)$ increases in both δ and N , and $\frac{k^*(N, \delta)}{N} \leq \frac{2}{3}$. This means that it is optimal to share information with more firms as the number of firms increase in the market and as firms offer closer substitutes, but that it is optimal to leave at least a third of firms uninformed.

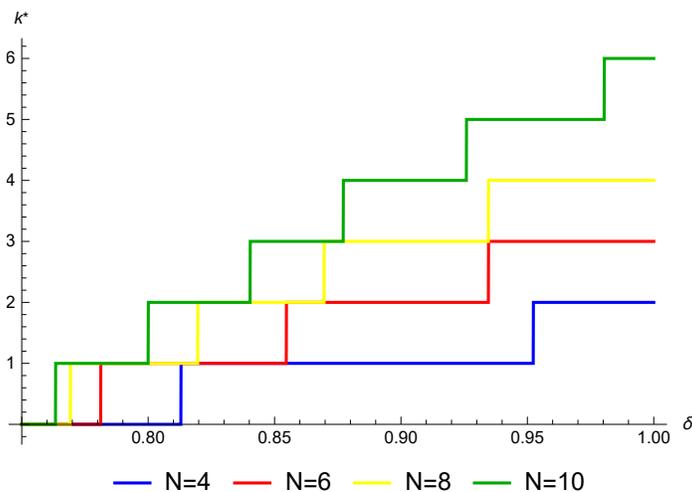


Figure 6: Consumer-optimal k as a function of δ for different market sizes.

7 Conclusion

This paper studies information disclosure in a setting where two competing firms face ex-ante information asymmetry about the level of demand. I examine the incentives of an informed firm to share its private information with a competitor in a market with product differentiation and price competition. I show that the informed firm can have incentives to fully disclose its private information even without receiving information in return, because it allows them to influence competitor pricing. When firms offer substitutes, they benefit from price correlation, which implies that it is optimal for the informed firm to fully reveal its private information to the uninformed firm. When firms offer complements, it is optimal for the informed firm to not share any private information, which reduces the expected profits of its competitor. Accordingly, it can be optimal for a designer with the objective of maximizing producer surplus or maintaining competition to intervene and force information disclosure.

Further, information disclosure also impacts consumers. Even though complete information disclosure can help firms, it hurts consumers. I find that a regulator with the objective of protecting consumers would either completely restrict information disclosure between firms or only allow private partial disclosure, determined by the degree of differentiation between products. If goods are sufficiently close substitutes, partial disclosure is optimal, because it increases arbitrage opportunities for consumers. The consumer optimal partial disclosure reveals low levels of demand and obfuscates high levels to the uninformed firm.

Moreover, the preferences for information disclosure between firms and consumers are not aligned. When firms offer substitutes, the optimal disclosure depends on the degree of substitution, which determines the effect of disclosure on consumers and firms. If firms offer sufficiently differentiated goods, no disclosure maximizes expected welfare. If firms offer sufficiently close substitutes, full disclosure is optimal. For intermediate levels of differentiation, partial disclosure is optimal. Since incentives for partial disclosure derive from consumers, the optimal partial disclosure also reveals low levels and obfuscates high levels to the uninformed firm. My results highlight the wide scope for potential intervention by regulators, depending on their objective function and product differentiation.

An important aspect not considered in this paper is the effect of information disclosure on firm entry and exit decisions. In particular, the informed firm may reduce its information disclosure, reducing its current profits, to increase its market share and profits in the future by inducing uninformed firms to exit the market. In this context, a regulator may have incentives to force information disclosure between firms in order to maintain the level of competition in the market, which indirectly may also benefit consumers.

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A Appendix

A.1 Useful results for the binary signal benchmark

In this section, I derive the optimal information structures for the benchmark case in which signals are restricted to be binary. Assume that the set of signals is binary and given by $S = \{s_L, s_H\}^2$. The information structure (S, x) with conditional distributions $x : \Theta \rightarrow \Delta(S)$ can be represented in matrix form where rows represent firm 1's signal realization and columns firm 2's signal realization. Define π as follows:

$\theta = \theta_L$	s_L	s_H
s_L	x_L	$1 - x_L$
s_H	0	0

$\theta = \theta_H$	s_L	s_H
s_L	0	0
s_H	$1 - x_H$	x_H

The set of feasible information structure, denoted by \mathcal{D} , is

$$\mathcal{D} := \{(x_L, x_H) \in [0, 1]^2 : x_L + x_H \geq 1\}.$$

Optimal pricing and equilibrium outcomes. Given the information structure (S, x) and conditional on the realization of signal s_i , firm i chooses $p_i(s_i) \geq 0$ to maximize her expected profits,

$$\max_{p_i(s_i) \geq 0} \Pi_i(p_i(s_i), p_{-i}(s_{-i})) = p_i(s_i) [\mathbb{E}[\theta | s_i] + b\mathbb{E}[p_{-i}(s_{-i}) | s_i] - ap_i(s_i)].$$

Equilibrium prices $(p_1^*(s_1), p_2^*(s_2))$ are the unique solution to:

$$\begin{aligned} \mathbb{E}[\theta | s_i] + b\mathbb{E}[p_j(s_j) | s_i] - 2ap_i(s_i) &= 0 \text{ for all } s_i \in S_i, i \in \{1, 2\} \text{ and } j \neq i \text{ where} \\ \mathbb{E}[\theta | s_i = s_\ell] &= \mathbb{P}(\theta = \theta_L | s_i = s_\ell)\theta_L + \mathbb{P}(\theta = \theta_H | s_i = s_\ell)\theta_H \text{ and} \\ \mathbb{E}[p_{-i}(s_{-i}) | s_i = s_\ell] &= \mathbb{P}(s_{-i} = s_L | s_i = s_\ell)p_{-i}(s_L) + \mathbb{P}(s_{-i} = s_H | s_i = s_\ell)p_{-i}(s_H). \end{aligned}$$

Consumer optimal disclosure. Assume that the designer's objective is to maximize expected consumer surplus. The optimal disclosure is determined by the relationship between goods, as stated in Lemma 11.

Lemma 11 *If the designer's objective is to maximize expected consumer surplus, partial disclosure is optimal if $\delta \in (\hat{c}, 1)$ and no disclosure is CS-optimal, otherwise.*

Proof. Lemma 11. First, full disclosure is never CS-optimal since expected consumer surplus is higher with no information disclosure than with full disclosure since

$$CS(x_L, 1 - x_L) - CS(1, 1) \geq \frac{\mu_L \mu_H (a^4 + b^4) (\theta_H - \theta_L)^2}{8a^3(2a - b)^2} \geq 0,$$

implying that either no or partial disclosure maximizes consumer surplus.

Second, I show that there exists $\hat{c} \in (0, 1)$ such that partial disclosure is optimal if $\delta \geq \hat{c}$ and no disclosure is optimal otherwise. Define $\Delta\mathbb{E}[CS](x)$ as the difference between the expected consumer surplus with no disclosure $\pi^N = (x_L, 1 - x_L)$ and the expected consumer surplus with disclosure (x_L, x_H) . The sign of $\Delta\mathbb{E}[CS](x)$ is determined by

$$\Phi(a, b, x) = f_1(a, b)\mathbb{V}[s_2] + f_2(a, b)\mu_L\mu_H(x_L + x_H - 1)^2 - f_3(a, b)\mathbb{E}[\mathbb{V}[s_2|\theta]]$$

where

$$f_1(a, b) = a^2(4a^2(6a + b) - b^2(18a + 7b))$$

$$f_2(a, b) = b^4(2a + b)$$

$$f_3(a, b) = a^2b^2(6a + 5b)$$

and $f_k(a, b) > 0$ for all $k \in \{1, 2, 3\}$, $\min\{f_1(a, b), f_3(a, b)\} > f_2(a, b)$ for all $a > b > 0$, $f_1(a, b) > f_3(a, b)$ if and only if $\delta < \hat{c} \approx 0.9$. This implies that

$$f_1(a, b)\mathbb{V}[s_2] > f_3(a, b)\mathbb{E}[\mathbb{V}[s_2|\theta]] \text{ if } \delta < \hat{c}$$

since $f_1(a, b) > f_3(a, b)$ and $\mathbb{V}[s_2] > \mathbb{E}[\mathbb{V}[s_2|\theta]]$. Thus, no disclosure maximizes the expected consumer surplus if $\delta \leq \hat{c}$. Otherwise, partial disclosure is CS-optimal since for all $\delta > \hat{c}$, there exists $x \in \mathcal{D}$ such that $\Phi(a, b, x) < 0$. ■

Welfare optimal disclosure. Assume that the designer's objective is to maximize expected welfare, defined as the sum of expected consumer surplus and expected firm profits.

Lemma 12 *Assume that the designer's objective is to maximize expected welfare. If*

1. $\delta \in (0, \tilde{c}_1]$, no disclosure is optimal
2. $\delta \in (\tilde{c}_1, \tilde{c}_2)$, partial disclosure is optimal.
3. $\delta \in [\tilde{c}_2, 1)$, full disclosure is optimal.

Proof. Lemma 12. First, define $\Delta\mathbb{E}[TS_1](x)$ as the difference in expected welfare with full disclosure x^F and a partial disclosure characterized by x . The sign of this difference is determined by

$$\rho_1(a, b, \mu, x) = f_4(a, b)\mathbb{V}[s_2] + f_5(a, b)\mathbb{E}[\mathbb{V}[s_2|\theta]] + \mu_L\mu_H f_6(a, b)(x_L + x_H - 1)^2$$

where

$$f_4(a, b) = 16a^5(3b - a), \quad f_5(a, b) = 4a^2b^2(5a^2 - b^2) \text{ and } f_6(a, b) = b^2(16a^4 - 12a^3b + a^2b^2 - b^4)$$

Note that $\rho_1(a, b, \mu, x) > 0$ for all $x \in \mathcal{D}$ if $\delta \geq \tilde{c}_2 \approx 0.31$ and there exists $x \in \mathcal{D}$ such that $\rho_1(a, b, \mu, x) < 0$ if $\delta < \tilde{c}_2$. Thus, full disclosure is optimal if $\delta \geq \tilde{c}_2$ and either partial or no disclosure is optimal otherwise.

Second, define $\Delta\mathbb{E}[TS_2](x)$ as the difference of expected welfare with no disclosure x^N and with partial disclosure x . The sign of $\Delta\mathbb{E}[TS_2](x)$ is determined by the sign of

$$\rho_2(a, b, \mu, x) = f_7(a, b)\mathbb{V}[s_2] - f_8(a, b)\mathbb{E}[\mathbb{V}[s_2|\theta]] - f_9(a, b)\mu_L\mu_H(x_L + x_H - 1)^2$$

where

$$f_7(a, b) = 4a^4(2a - 5b), \quad f_8(a, b) = 12a^2b^2(2a + b) \text{ and } f_9(a, b) = b^2(22a^3 + 5a^2b - b^2(2a + b))$$

and $f_k(a, b) > 0$ for all $k \in \{7, 8, 9\}$ since $a > b > 0$. Note that $\rho_2(a, b, \mu, x) \geq 0$ for all $x \in \mathcal{D}$ if $\delta \leq \tilde{c}_1 \approx 0.29$ and for $\delta > \tilde{c}_1$, there exists $x \in \mathcal{D}$ such that $\rho_2(a, b, \mu, x) < 0$. Thus, no disclosure is optimal if $\delta \leq \tilde{c}_1$ and either full or partial disclosure is optimal otherwise. In summary, no disclosure is optimal if $\delta \leq \tilde{c}_1$, partial disclosure is optimal if $\delta \in (\tilde{c}_1, \tilde{c}_2]$ and full disclosure is optimal if $\delta > \tilde{c}_2$. ■

A.2 Proofs

A.2.1 Preliminary results: proofs

Proof. Lemma 1. The pricing game is a smooth concave game since $\Pi_i((\cdot, p_{-i}); \theta) : \mathbb{R}_+ \rightarrow \mathbb{R}$ is concave and continuously differentiable for each $p_{-i} \in \mathbb{R}_+$ since the demand is linear in p_{-i} . Define the payoff gradient as

$$\nabla\Pi(\mathbf{p}, \theta) := \left(\frac{\partial\Pi_i((p_i, p_{-i}); \theta)}{\partial p_i} \right)_{i \in \{1, 2\}},$$

where firm i 's ex-post payoff function is given by $\Pi_i((p_i, p_{-i}); \theta) = p_i(\theta - ap_i + bp_{-i})$. Then, the payoff gradient, given by

$$\nabla\Pi(\mathbf{p}, \theta) = (\theta + bp_{-i} - 2ap_i)_{i \in \{1, 2\}},$$

is continuously differentiable. The Jacobian matrix of the payoff gradient, given by

$$F_{\nabla\Pi}(\mathbf{p}, \theta) := \begin{pmatrix} \frac{\partial^2\Pi_1((p_1, p_2); \theta)}{\partial p_1^2} & \frac{\partial^2\Pi_1((p_1, p_2); \theta)}{\partial p_1 \partial p_2} \\ \frac{\partial^2\Pi_2((p_2, p_1); \theta)}{\partial p_1 \partial p_2} & \frac{\partial^2\Pi_2((p_2, p_1); \theta)}{\partial p_2^2} \end{pmatrix} = \begin{pmatrix} -2a & b \\ b & -2a \end{pmatrix},$$

is negative definite because $-2a < 0$ and $4a^2 - b^2 > 0$ since $a > |b|$. This implies that the payoff gradient $\nabla\Pi(\mathbf{p}, \theta)$ is strictly monotone by Lemma 4 from Ui (2016). Furthermore, since for all \mathbf{p} , there exists $c > 0$ such that

$$\mathbf{p}^T F_{\nabla\Pi}(\mathbf{p}, \theta)\mathbf{p} < -c\mathbf{p}^T \mathbf{p},$$

the payoff gradient is also strongly monotone by the same lemma. Then, the uniqueness of the Bayesian Nash equilibrium of the pricing game follows from Proposition 1 from Ui (2016), which states that if the payoff gradient is strictly monotone, the Bayesian game as at most one Bayesian Nash equilibrium. The existence of a unique Bayesian Nash equilibrium follows from Proposition 2 from Ui (2016). ■

Proof. Lemma 2. First, I show that the set of BCE is a subset of $\cup_{(S, \psi)} \mathcal{E}(S, \psi)$. Assume $\sigma \in BCE$. Then, σ satisfies

$$\sum_{\theta \in \Theta} \mu_\theta \int_{p_{-i} \in [0, \frac{\theta_H}{a-b}]} \Pi_i((p_i, p_{-i}), \theta) d\sigma((p_i, p_{-i})|\theta) \geq \sum_{\theta \in \Theta} \mu_\theta \int_{p_{-i} \in [0, \frac{\theta_H}{a-b}]} \Pi_i((p'_i, p_{-i}), \theta) d\sigma((p_i, p_{-i})|\theta) \quad (7)$$

for all $p_i \in \text{supp } \sigma$, $p'_i \in [0, \frac{\theta_H}{a-b}]$ and $i \in \{1, 2\}$.

Consider an information structure $([0, \frac{\theta_H}{a-b}]^2, \psi^*)$ where $[0, \frac{\theta_H}{a-b}]^2$ is the set of signal realizations and $\psi^* : \Theta \rightarrow \Delta([0, \frac{\theta_H}{a-b}]^2)$ coincides with σ , i.e. $\sigma = \psi^*$. Let

$$\beta_i^*(p_i|p'_i) = \begin{cases} 1 & \text{if } p_i = p'_i \\ 0 & \text{otherwise} \end{cases}$$

be the obedient strategy. Then, the right-hand side of (7) can be written as

$$\begin{aligned} & \sum_{\theta \in \Theta} \mu_\theta \int_{p_{-i} \in [0, \frac{\theta_H}{a-b}]} \Pi_i((p'_i, p_{-i}), \theta) d\sigma((p_i, p_{-i})|\theta) \\ &= \sum_{\theta \in \Theta} \mu_\theta \int_{p_{-i} \in [0, \frac{\theta_H}{a-b}]} \Pi_i((p'_i, p_{-i}), \theta) d\psi^*((p_i, p_{-i})|\theta) \\ &= \sum_{\theta \in \Theta} \mu_\theta \int_{s_{-i} \in [0, \frac{\theta_H}{a-b}]} \int_{p_{-i} \in [0, \frac{\theta_H}{a-b}]} \Pi_i((p'_i, p_{-i}); \theta) d\beta_{-i}^*(p_{-i}|s_{-i}) d\psi^*((s_i, s_{-i})|\theta) \end{aligned}$$

The first equality holds by definition of ψ^* . The second equality holds by definition of the obedient strategy and Fubini's theorem since, fixing θ , $\Pi_i((p_i, p_{-i}); \theta)$ is σ -integrable because $\Pi_i|\theta : [0, \frac{\theta_H}{a-b}]^2 \rightarrow \mathbb{R}_+$ is a bounded and continuous real-valued function on a compact set.²³ Hence, the BNE incentive-compatibility constraints are implied by the BCE obedience

²³See theorem 11.27 from Aliprantis and Border (2013) where the condition of theorem are satisfied by Proposition 3.3 and Theorem 4.4 from Royden (1968)

constraints. This, in turn, implies that if $\sigma \in BCE$, then σ is also a BNE of the game. Thus, the set of BCE is a subset of the set of BNE of the game.

Second, I show that $\cup_{(S,\psi)} \mathcal{E}(S, \psi)$ is a subset of BCE. Consider a BNE composed by an information structure $(\hat{S}, \hat{\psi})$ with $\hat{\psi} : \Theta \rightarrow \Delta(S)$ and measurable behavioral strategies $(\hat{\beta}_i, \hat{\beta}_{-i})$.²⁴ Given the behavioral strategies $(\hat{\beta}_i, \hat{\beta}_{-i})$, define $\hat{\beta} : S \rightarrow \Delta\left(\left[0, \frac{\theta_H}{a-b}\right]^2\right)$ as the joint measure. Let $\hat{\sigma} : \Theta \rightarrow \Delta\left(\left[0, \frac{\theta_H}{a-b}\right]^2\right)$ be the composition of $\hat{\psi}$ and $\hat{\beta}$, defined as $\hat{\sigma} = \hat{\beta} \circ \hat{\psi}$. Then, by definition $\hat{\sigma} \in \cup_{(S,\psi)} \mathcal{E}(S, \psi)$. The definition of BNE implies that $(\hat{S}, \hat{\psi})$ and $\hat{\beta}$ satisfy:

$$\begin{aligned} & \sum_{\theta \in \Theta} \mu_\theta \int_{\hat{S}_{-i}} \int_{p_{-i} \in \left[0, \frac{\theta_H}{a-b}\right]} \Pi_i((p_i, p_{-i}); \theta) d\hat{\beta}_{-i}(p_{-i}|s_{-i}) d\hat{\psi}((s_i, s_{-i})|\theta) \\ & \geq \sum_{\theta \in \Theta} \mu_\theta \int_{\hat{S}_{-i}} \int_{p_{-i} \in \left[0, \frac{\theta_H}{a-b}\right]} \Pi_i((p'_i, p_{-i}); \theta) d\hat{\beta}_{-i}(p_{-i}|s_{-i}) d\hat{\psi}((s_i, s_{-i})|\theta) \end{aligned} \quad (8)$$

for all $p'_i \in \left[0, \frac{\theta_H}{a-b}\right]$, $s \in S$ and $i \in \{1, 2\}$. Integrating both sides of the BNE incentive-compatibility constraint, we have

$$\begin{aligned} & \sum_{\theta \in \Theta} \mu_\theta \int_{\hat{S}_i} \int_{\hat{S}_{-i}} \int_{p_{-i} \in \left[0, \frac{\theta_H}{a-b}\right]} \Pi_i((p_i, p_{-i}); \theta) d\hat{\beta}_i(p_i|s_i) d\hat{\beta}_{-i}(p_{-i}|s_{-i}) d\hat{\psi}(s|\theta) \\ & \geq \sum_{\theta \in \Theta} \mu_\theta \int_{\hat{S}_i} \int_{\hat{S}_{-i}} \int_{p_{-i} \in \left[0, \frac{\theta_H}{a-b}\right]} \Pi_i((p'_i, p_{-i}); \theta) d\hat{\beta}_i(p_i|s_i) d\hat{\beta}_{-i}(p_{-i}|s_{-i}) d\hat{\psi}(s|\theta) \end{aligned}$$

Then, (8) implies that

$$\sum_{\theta \in \Theta} \mu_\theta \int_{p_{-i} \in \left[0, \frac{\theta_H}{a-b}\right]} \Pi_i((p_i, p_{-i}), \theta) d\sigma((p_i, p_{-i})|\theta) \geq \sum_{\theta \in \Theta} \mu_\theta \int_{p_{-i} \in \left[0, \frac{\theta_H}{a-b}\right]} \Pi_i((p'_i, p_{-i}), \theta) d\sigma((p_i, p_{-i})|\theta)$$

■

Proof. Lemma 3. Consider a distribution $\sigma \in \cup_{(S,\psi)} \mathcal{E}(S, \psi)$. Lemma 2 implies that $\sigma \in BCE$. Consider the recommendation mechanism $(\left[0, \frac{\theta_H}{a-b}\right]^2, \psi_\sigma)$ where $\psi_\sigma = \sigma$ for all $(p_1, p_2) \in \left[0, \frac{\theta_H}{a-b}\right]^2$ and $\theta \in \Theta$ and the obedient behavioral strategy

$$\beta_i^*(p_i|p'_i) = \begin{cases} 1 & \text{if } p_i = p'_i \\ 0 & \text{otherwise} \end{cases}.$$

²⁴Behavioral strategies $\beta_i : S_i \rightarrow \Delta\left(\left[0, \frac{\theta_H}{a-b}\right]\right)$ for all $i \in \{1, 2\}$ are defined as a regular conditional probabilities as defined in Appendix C from Bass (2011).

The interim expected payoff of firm i when firm $-i$ follows β_{-i}^* is

$$\begin{aligned}
& \sum_{\theta \in \Theta} \mu_{\theta} \int_{S_{-i}} \int_0^{\frac{\theta_H}{a-b}} \Pi_i((p'_i, p_{-i}); \theta) d\beta_{-i}^*(p_{-i}|p'_i) d\psi_{\sigma}((p_i, p'_{-i})|\theta) \\
&= \sum_{\theta \in \Theta} \mu_{\theta} \int_0^{\frac{\theta_H}{a-b}} \Pi_i((p'_i, p_{-i}); \theta) d\psi_{\sigma}((p_i, p_{-i})|\theta) \\
&= \sum_{\theta \in \Theta} \mu_{\theta} \int_0^{\frac{\theta_H}{a-b}} \Pi_i((p'_i, p_{-i}); \theta) d\sigma((p_i, p_{-i})|\theta) \tag{9}
\end{aligned}$$

for all i . Hence, the definition of BCE and (9) imply

$$\sum_{\theta \in \Theta} \mu_{\theta} \int_0^{\frac{\theta_H}{a-b}} \Pi_i((p_i, p_{-i}); \theta) d\psi_{\sigma}((p_i, p_{-i})|\theta) \geq \sum_{\theta \in \Theta} \mu_{\theta} \int_0^{\frac{\theta_H}{a-b}} \Pi_i((p'_i, p_{-i}); \theta) d\psi_{\sigma}((p_i, p_{-i})|\theta)$$

for all $p'_i \in [0, \frac{\theta_H}{a-b}]$ and i . The distribution of prices conditional on the state θ under β^* and $([0, \frac{\theta_H}{a-b}]^2, \sigma)$ is $\psi_{\sigma} = \sigma$. Thus, $\sigma \in \mathcal{E}([0, \frac{\theta_H}{a-b}]^2, \sigma)$. ■

Lemma 13 *The support of the distribution $\sigma((p_1, p_2)|\theta)$ is a subset of $[p^F(\theta_L), p^F(\theta_H)]^2$ for all $\theta \in \Theta$, where $p^F(\theta)$ is the equilibrium price with full disclosure when the state θ is realized.*

Proof. Lemma 13. The minimum and maximum price in any equilibrium is charged when both firms know that the state is low and that the state is high, respectively. That is, the highest and lowest equilibrium prices occur with full disclosure. Under full disclosure σ^F , both firms learn the state. Let $p^F(\theta)$ be the equilibrium price under full disclosure when the state is θ , where

$$p^F(\theta_L) = \frac{\theta_L}{(2a-b)} \text{ and } p^F(\theta_H) = \frac{\theta_H}{(2a-b)}$$

Hence, any obedient recommendation mechanism must recommend prices in the set of feasible equilibrium prices denoted by $[p^F(\theta_L), p^F(\theta_H)]^2$. ■

Proof. Lemma 4. The set of BCE is the collection of distributions $\sigma : \Theta \rightarrow \Delta([p^F(\theta_L), p^F(\theta_H)]^2)$ such that

- i) $\sigma((p_1, p_2)|\theta) \geq 0$ for all $(p_1, p_2) \in [p^F(\theta_L), p^F(\theta_H)]^2$ and $\theta \in \Theta$,
- ii) $\int d\sigma((p_1, p_2)|\theta) = 1$ for all $\theta \in \Theta$ and
- iii) $\sum_{\theta \in \Theta} \mu_{\theta} \int_{p_{-i} \in \mathbb{R}_+} \Pi_i((p_i, p_{-i}), \theta) d\sigma((p_i, p_{-i})|\theta) \geq \sum_{\theta \in \Theta} \mu_{\theta} \int_{p_{-i} \in \mathbb{R}_+} \Pi_i((p'_i, p_{-i}), \theta) d\sigma((p_i, p_{-i})|\theta)$
for all $p_i \in \text{supp } \sigma$, $p'_i \in \mathbb{R}_+$ and $i \in \{1, 2\}$.

First, Theorem A from Stinchcombe (2011) establishes the existence of Correlated equilibrium in games in which players receive private signals and then simultaneously choose actions from compact sets. Formally, consider a game in which the set of players I is finite and for each i , the type ω_i belongs to the measure space $(\Omega_i, \mathcal{F}_i)$. Each player i simultaneously chooses an action from a compact set A_i and denote by Δ_i the set of countably additive Borel probabilities in A_i , with the weak* topology. Let $\mathbb{B}_i(\mathcal{F}_i)$ be the set of i 's behavioral strategies, defined as the \mathcal{F}_i -measurable functions from Ω_i to Δ_i . Given a vector $b \in \mathbb{B} := \times_i \mathbb{B}_i(\mathcal{F}_i)$, player i 's expected utility if b is played is defined by

$$u_i^P(b) = \int_{\Omega} \langle u_i(\omega), \times_i b_i(\omega) \rangle P(d\omega)$$

where $\langle f, \nu \rangle := \int_A f(a) \nu(da)$ for $f : A \rightarrow \mathbb{R}$ and Borel probabilities ν , and $\times_i b_i$ is the product probability on A having b_i as the marginal. $(\mathbb{B}_i(\mathcal{F}_i), u_i^P)_{i \in I}$ denotes the normal form game. Then, Theorem A shows that all games $(\mathbb{B}_i(\mathcal{F}_i), u_i^P)_{i \in I}$ have correlated equilibria.

In the pricing game, two firms simultaneously choose a price to maximize their expected equilibrium profits,

$$\sum_{\theta \in \Theta} \mu_{\theta} \int_{p_{-i} \in \mathbb{R}_+} \Pi_i((p_i, p_{-i}), \theta) d\sigma((p_i, p_{-i})|\theta).$$

Note that $p_i \in [0, \frac{\theta_H}{a-b}]$ for all $i \in \{1, 2\}$. Hence, firms simultaneously choose prices from compact sets. Thus, this result implies that the set of BCE is non-empty.

Second, the set of BCE is the collection of distributions

$$\sigma : \Theta \rightarrow \Delta([p^F(\theta_L), p^F(\theta_H)]^2),$$

which corresponds to the set of all probability measures on $[p^F(\theta_L), p^F(\theta_H)]^2$ for each $\theta \in \Theta$ where Θ is finite. Then, the set of BCE is compact since $[p^F(\theta_L), p^F(\theta_H)]^2$ is compact in the weak* topology, by Theorem 15.11 from Aliprantis and Border (2013).

The designer's objectives are

- i) Informed firm optimal : $\sum_{\theta \in \Theta} \mu_{\theta} \int \Pi_1((p_1, p_2), \theta) d\sigma((p_1, p_2)|\theta)$
- ii) PS optimal : $\sum_{i \in \{1, 2\}} \sum_{\theta \in \Theta} \mu_{\theta} \int \Pi_i((p_i, p_{-i}), \theta) d\sigma((p_i, p_{-i})|\theta)$
- iii) CS-optimal : $\frac{1}{2a} \sum_{i \in \{1, 2\}} \sum_{\theta \in \Theta} \mu_{\theta} \int q_i((p_i, p_{-i}); \theta)^2 d\sigma((p_i, p_{-i})|\theta)$

and

$$\begin{aligned}
iv) \text{ Welfare-optimal : } & \sum_{i \in \{1,2\}} \sum_{\theta \in \Theta} \mu_\theta \int \Pi_i((p_i, p_{-i}), \theta) d\sigma((p_i, p_{-i})|\theta) \\
& + \frac{1}{2a} \sum_{i \in \{1,2\}} \sum_{\theta \in \Theta} \mu_\theta \int q_i((p_i, p_{-i}); \theta)^2 d\sigma((p_i, p_{-i})|\theta)
\end{aligned}$$

Third, the continuity of all objective functions in the weak* topology follows from Corollary 15.7 from Aliprantis and Border since because both $\Pi_i((p_i, p_{-i}), \theta)$ and $q_i((p_i, p_{-i}), \theta)$ are continuous and bounded functions. Hence, the integral $\int \Pi_i d\sigma((p_i, p_{-i})|\theta)$ and $\int q_i^2 d\sigma((p_i, p_{-i})|\theta)$ is continuous in σ . Thus, the designer's problem is to maximize a continuous objective function in a compact set. The existence of a solution is guaranteed by the Weierstrass extreme value theorem. ■

A.2.2 Informed firm optimal disclosure: proofs

Proof. Proposition 1. The fully disclosing information structure recommends prices $(p^F(\theta), p^F(\theta))$ with probability 1 for all $\theta \in \Theta$.

Full disclosure is optimal for the informed firm if her expected equilibrium payoffs with full disclosure exceed her expected equilibrium payoffs induced by any other obedient recommendation mechanism. That is,

$$\sum_{\theta \in \Theta} \mu_\theta \Pi_1((p^F(\theta), p^F(\theta)); \theta) \geq \sum_{\theta \in \Theta} \mu_\theta \int \Pi_1((p_1, p_2); \theta) d\sigma((p_1, p_2)|\theta) \quad (10)$$

for all $\sigma : \Theta \rightarrow \Delta([p^F(\theta), p^F(\theta)]^2)$ that satisfy the obedience constraints and p_1 . The obedience constraints requires that given p_2 , p_1 must be a best response for firm 1. Then, for all recommendation mechanism σ that satisfy the obedience constraints, the RHS of (10) is given by

$$\begin{aligned}
\sum_{\theta \in \Theta} \mu_\theta \int \Pi_1((p_1, p_2); \theta) d\sigma((p_1, p_2)|\theta) &= \sum_{\theta \in \Theta} \mu_\theta \int \frac{\theta + b\mathbb{E}_\sigma[p_2|\theta]}{2a} \left[\theta + bp_2 - \frac{\theta + b\mathbb{E}_\sigma[p_2|\theta]}{2} \right] d\sigma(p_1, p_2|\theta) \\
&= a\mathbb{E}_\mu \left[\left(\frac{\theta + b\mathbb{E}_\sigma[p_2|\theta]}{2a} \right)^2 \right] = a\mathbb{E}_\mu [p_1^\sigma(\theta)^2]
\end{aligned}$$

The first equality holds since $\Pi_1((p_1, p_2); \theta) = p_1(\theta + bp_2 - ap_1)$ and since firm 1's best response, denoted by $p_1^\theta(\theta)$, is

$$p_1^\theta(\theta) = \frac{\theta + b\mathbb{E}_\sigma[p_2|\theta]}{2a}.$$

When firms offer substitutes, firm 1's expected equilibrium profit is an increasing and convex function of the expected equilibrium price p_2 . Then, Jensen's inequality implies

that maximizing expected equilibrium profits is equivalent to maximizing the distance between the expected equilibrium prices set by firm 2, $\mathbb{E}_\sigma[p_2|\theta]$ (or, equivalently, by maximizing the distance between $p_1^\sigma(\theta)$). When firms offer substitutes, Lemma 13 shows that $\sup \sigma((p_1, p_2)|\theta) \in [p^F(\theta_L), p^F(\theta_H)]^2$ for all $\theta \in \Theta$. Hence, recommending $(p^F(\theta_L), p^F(\theta_L))$ in the low state and $(p^F(\theta_H), p^F(\theta_H))$ in the high state maximizes expected equilibrium profit which implies that full disclosure is optimal for the informed firm.²⁵ ■

A.2.3 Consumer optimal disclosure: proofs

Proof. Proposition 2. Lemma 11 shows that full disclosure is never optimal for consumers. Consider instead any partial disclosure policy σ and define $\sigma(s_2|\theta)$ the distribution of price recommendation p_2 conditional on the state θ . Expected consumer surplus, denoted by $\mathbb{E}_{(\mu, \sigma)}[CS((p_1, p_2); \theta)]$, is

$$\mathbb{E}_{(\mu, \sigma)}[CS((p_1, p_2); \theta)] = \frac{1}{2a} \sum_{\theta \in \Theta} \mu_\theta \left[\int (\theta + bp_2 - ap_1)^2 d\sigma((p_1, p_2)|\theta) + \int (\theta + bp_1 - ap_2)^2 d\sigma((p_1, p_2)|\theta) \right] \quad (11)$$

where, in the unique BNE, p_1 satisfies

$$p_1 = \frac{1}{2a} \left[\theta + b \int p_2 d\sigma(p_2|\theta) \right].$$

Substituting this expression in (11), expected consumer surplus can be written as

$$\mathbb{E}_{(\mu, \sigma)}[CS((p_1, p_2); \theta)] = \frac{1}{2a} \mathbb{E}_\mu \left[\mathbb{E}_\sigma \left[\left(\frac{\theta}{2} + b \left(p_2 - \frac{1}{2} \mathbb{E}_\sigma[p_2|\theta] \right) \right)^2 \middle| \theta \right] \right] + \frac{1}{2a} \mathbb{E}_\mu \left[\mathbb{E}_\sigma \left[\left[\theta \left(1 + \frac{b}{2a} \right) + \frac{b^2}{2a} \mathbb{E}_\sigma[p_2|\theta] - ap_2 \right]^2 \middle| \theta \right] \right]$$

Define $\Delta \mathbb{E}[CS](\sigma)$ as the difference in expected consumer surplus with partial and no disclosure. This difference is given by

$$\Delta \mathbb{E}[CS](\sigma) = \frac{a}{2} (\delta^2 + 1) \mathbb{V}_{(\mu, \sigma)}[p_2] - \left[\left(1 - \frac{\delta^2}{2} \right) \left(\frac{\delta}{2} + 1 \right) - \frac{\delta}{4} \right] Cov_{(\mu, \sigma)}(\theta, p_2) - \frac{b\delta}{8} (7 - \delta^2) \mathbb{V}_\mu[\mathbb{E}_\sigma[p_2|\theta]],$$

where the equality holds by the law of iterated expectations, the definition of variance, conditional variance and covariance and the law of total variance. Hence, the difference in expected consumer surplus, $\Delta \mathbb{E}[CS](\sigma)$, is a continuous function of δ . This difference is also a strictly increasing function of δ . In particular, Lemma 11 shows that $\Delta \mathbb{E}[CS](\sigma)$ converges to a positive number as $\delta \rightarrow 1$ which, in turn, implies that

$$b > \frac{2Cov_{(\mu, \sigma)}[\theta, p_2]}{4\mathbb{V}_{(\mu, \sigma)}[p_2] - 3\mathbb{V}_\mu[\mathbb{E}_\sigma[p_2|\theta]]},$$

²⁵That is, to maximize the expectation of a quadratic function in an interval, it is necessary to put all mass on the extremes of such interval.

and this condition ensures that $\Delta\mathbb{E}[CS](\sigma)$ is a strictly increasing function of δ .

First, if $\delta \rightarrow 0$, the expected consumer surplus with partial and no disclosure converge to

$$\Delta\mathbb{E}[CS](\sigma) \xrightarrow{\delta \rightarrow 0} \frac{a}{2}V_{(\mu,\sigma)}[p_2] - Cov_{(\mu,\sigma)}[\theta, \mathbb{E}[p_2|\theta]] = -Cov_{(\mu,\sigma)}\left[\theta - \frac{a}{2}p_2, p_2\right].$$

The equality holds by properties of covariance and since the covariance between θ and $\mathbb{E}_\sigma[p_2|\theta]$ equals the covariance between θ and p_2 . The price p_2 is an increasing function of θ since the state is a positive demand shifter and

$$\frac{\partial p_2}{\partial \theta} \leq \frac{1}{2a-b} \leq \frac{2}{a}$$

since $a > b > 0$. Then, the covariance between $\theta - \frac{a}{2}p_2$ and p_2 is the covariance between two increasing functions of θ . Hence, this covariance is positive, which implies that $\Delta\mathbb{E}[CS](\sigma)$ converges to a negative number when $\delta \rightarrow 0$.

Second, Lemma 11 shows that partial disclosure yields higher expected consumer surplus than no disclosure if $\delta \in (\hat{c}, 1)$. That is, $\Delta\mathbb{E}[CS](\sigma) > 0$ for all $\delta \in (\hat{c}, 1)$. Hence, the Intermediate Value theorem implies that there exists $\hat{\alpha} \in (0, \hat{c}]$ such that $\Delta\mathbb{E}[CS](\sigma) = 0$ when $\delta = \hat{\alpha}$. Moreover, since $\Delta\mathbb{E}[CS](\sigma)$ is strictly increasing in δ , partial disclosure is optimal for all $\delta \in (\hat{\alpha}, 1)$ where $\hat{\alpha} \in (0, \hat{c}]$ and no disclosure is optimal otherwise. ■

Lemma 14 *Assume that σ is partially informative and $\sigma(p_2|\theta)$ is degenerated, placing all mass on $\hat{p} \in [p_L^F, p_H^F]$. For any obedient σ , $\text{supp } \sigma(p_2|\theta') = \{\hat{p}, \hat{p}'\}$ for all $\theta \neq \theta'$.*

Proof. Lemma 14. The recommendation mechanism σ is not fully informative. First, I show that $\hat{p} \in \text{supp } \sigma(p_2|\theta')$. Suppose not. Then, $\text{supp } \sigma|\theta \cap \text{supp } \sigma|\theta' = \emptyset$ which implies that price recommendations fully reveal the state. However, this contradicts the assumption that σ is partially informative. Hence, $\hat{p} \in \text{supp } \sigma(p_2|\theta')$.

Second, I show that the support of $\sigma(p_2|\theta')$ is binary. Firm i 's obedience constraint is

$$\sum_{\theta \in \Theta} \mu_\theta \int_{p_{-i} \in \mathbb{R}_+} \Pi_i((p_i, p_{-i}), \theta) d\sigma((p_i, p_{-i})|\theta) \geq \sum_{\theta \in \Theta} \mu_\theta \int_{p_{-i} \in \mathbb{R}_+} \Pi_i((p'_i, p_{-i}), \theta) d\sigma((p_i, p_{-i})|\theta)$$

for all i , $p_i \in \text{supp } \sigma$ and $p'_i \in [p_L^F, p_H^F]$. The left-hand side of the uninformed firm obedience constraint can be simplified as follows:

$$\begin{aligned} \sum_{\theta \in \Theta} \mu_\theta \int_{p_1 \in [p_L^F, p_H^F]} p_2 (\theta + bp_1 - ap_2) d\sigma((p_1, p_2)|\theta) &= \sum_{\theta \in \Theta} \mu_\theta \int_{p_1 \in [p_L^F, p_H^F]} p_2 \left[\theta + b \left(\frac{\theta + b\mathbb{E}_\sigma[p_2|\theta]}{2a} \right) - ap_2 \right] d\sigma((p_1, p_2)|\theta) \\ &= \sum_{\theta \in \Theta} \mu_\theta p_2 \left[\theta + b \left(\frac{\theta + b\mathbb{E}_\sigma[p_2|\theta]}{2a} \right) - ap_2 \right] \int_{p_1 \in [p_L^F, p_H^F]} d\sigma((p_1, p_2)|\theta) \\ &= \sum_{\theta \in \Theta} \mu_\theta p_2 \left[\theta + b \left(\frac{\theta + b\mathbb{E}_\sigma[p_2|\theta]}{2a} \right) - ap_2 \right] \sigma(p_2|\theta) \end{aligned}$$

The first equality holds by the best response function of firm 1. The last equality holds since $\int_{p_1 \in [p_L^F, p_H^F]} d\sigma((p_1, p_2)|\theta) = \sigma(p_2|\theta)$. Hence, the uninformed firm obedience constraint is

$$\sum_{\theta \in \Theta} \mu_\theta p_2 \left[\theta + b \left(\frac{\theta + b\mathbb{E}_\sigma[p_2|\theta]}{2a} \right) - ap_2 \right] \sigma(p_2|\theta) \geq \sum_{\theta \in \Theta} \mu_\theta p'_2 \left[\theta + b \left(\frac{\theta + b\mathbb{E}_\sigma[p_2|\theta]}{2a} \right) - ap'_2 \right] \sigma(p_2|\theta)$$

for all $p_2 \in \text{supp } \sigma(p_2|\theta)$ and $p'_2 \in [p_L^F, p_H^F]$. The obedience constraint for $p_2 = \hat{p}$ is

$$\begin{aligned} & \mu_{\theta'} \sigma(\hat{p}|\theta') \hat{p} \left[\theta' + b \left(\frac{\theta' + b\mathbb{E}_\sigma[p_2|\theta']}{2a} \right) - a\hat{p} \right] + \mu_{\theta} \hat{p} \left[\theta + b \left(\frac{\theta + b\hat{p}}{2a} \right) - a\hat{p} \right] \\ & \geq \mu_{\theta'} \sigma(\hat{p}|\theta') p'_2 \left[\theta' + b \left(\frac{\theta' + b\mathbb{E}_\sigma[p_2|\theta']}{2a} \right) - ap'_2 \right] + \mu_{\theta} p'_2 \left[\theta + b \left(\frac{\theta + b\hat{p}}{2a} \right) - ap'_2 \right] \end{aligned}$$

for all $p'_2 \in [p_L^F, p_H^F]$. Similarly, the obedience constraint of σ for $p_2 \neq \hat{p}$ is

$$p_2 \left[\theta' + b \left(\frac{\theta' + b\mathbb{E}_\sigma[p_2|\theta']}{2a} \right) - ap_2 \right] \geq p'_2 \left[\theta' + b \left(\frac{\theta' + b\mathbb{E}_\sigma[p_2|\theta']}{2a} \right) - ap'_2 \right] \quad (12)$$

for all $p'_2 \in [p_L^F, p_H^F]$. The uninformed firm's profits are strictly concave in p_2 which implies there exists a unique $\hat{p}' \in [p_L^F, p_H^F]$ that satisfies (12) and $\hat{p}' \neq \hat{p}$. Hence, the support of $\hat{\sigma}|\theta'$ is binary and given by $\{\hat{p}, \hat{p}'\}$. ■

Lemma 15 *Assume that σ is partially informative and $\sigma(p_2|\theta_H)$ is degenerated, placing all mass on $\hat{p}_H \in [p_L^F, p_H^F]$. For any obedient σ , $\text{supp } \sigma(p_2|\theta_L) = \{\hat{p}_L, \hat{p}_H\}$ where $\lambda = \sigma(\hat{p}_L|\theta_L)$,*

$$\begin{aligned} \hat{p}_L &= \frac{4a^2[1 - \mu_L\lambda]\theta_L + b^2\mu_H[(1 - \lambda)\theta_H - \theta_L]}{(2a - b)[4a^2(1 - \mu_L\lambda) - b^2\mu_H\lambda]} \text{ and} \\ \hat{p}_H &= \frac{4a^2[\mu_H\theta_H + \mu_L(1 - \lambda)\theta_L] - b^2\mu_H\lambda\theta_H}{(2a - b)[4a^2(1 - \mu_L\lambda) - b^2\mu_H\lambda]} \end{aligned}$$

Proof. Lemma 15. Lemma 14 implies that the support of $\sigma(p_2|\theta_L)$ is binary and given by $\{\hat{p}_L, \hat{p}_H\}$ if the support of $\sigma(p_2|\theta_H)$ is degenerated and given by \hat{p}_H . Define $\sigma(\hat{p}_L|\theta_L) = 1 - \sigma(\hat{p}_H|\theta_L) := \lambda \in (0, 1)$. By definition,

$$\mathbb{E}_\sigma[p_2|\theta_L] = \lambda\hat{p}_L + (1 - \lambda)\hat{p}_H \text{ and } \mathbb{E}_\sigma[p_2|\theta_H] = \hat{p}_H.$$

Then, taking $\mathbb{E}_\sigma[p_2|\theta_L]$ and $\mathbb{E}_\sigma[p_2|\theta_H]$ as given, \hat{p}_L and \hat{p}_H are characterized by

$$\begin{aligned} \hat{p}_L &= \arg \max_{p_2} p_2 \left[\theta_L + b \left(\frac{\theta_L + b\mathbb{E}_\sigma[p_2|\theta_L]}{2a} \right) - ap_2 \right] \\ \hat{p}_H &= \arg \max_{p_2} \mu_L(1 - \lambda)p_2 \left[\theta_L + b \left(\frac{\theta_L + b\mathbb{E}_\sigma[p_2|\theta_L]}{2a} \right) - ap_2 \right] + \mu_H p_2 \left[\theta_H + b \left(\frac{\theta_H + b\mathbb{E}_\sigma[p_2|\theta_H]}{2a} \right) - ap_2 \right] \end{aligned}$$

The first order conditions of the previous maximization problems are

$$\begin{aligned}\hat{p}_L &= \frac{1}{2a} \left[\theta_L + \frac{b}{2a} (\theta_L + b\mathbb{E}_\sigma[p_2|\theta_L]) \right] \\ \hat{p}_H &= \frac{\mu_L(1-\lambda) [\theta_L + \frac{b}{2a} (\theta_L + b\mathbb{E}_\sigma[p_2|\theta_L])] + \mu_H [\theta_H + \frac{b}{2a} (\theta_H + b\mathbb{E}_\sigma[p_2|\theta_H])]}{2a(\mu_L(1-\lambda) + \mu_H)}\end{aligned}$$

Using the definition of $\mathbb{E}_\sigma[p_2|\theta_L]$ and $\mathbb{E}_\sigma[p_2|\theta_H]$, we have that \hat{p}_L and \hat{p}_H are given by

$$\begin{aligned}\hat{p}_L &= \frac{4a^2[1 - \mu_L\lambda]\theta_L + b^2\mu_H[(1 - \lambda)\theta_H - \theta_L]}{(2a - b)[4a^2(1 - \mu_L\lambda) - b^2\mu_H\lambda]} \text{ and} \\ \hat{p}_H &= \frac{4a^2[\mu_H\theta_H + \mu_L(1 - \lambda)\theta_L] - b^2\mu_H\lambda\theta_H}{(2a - b)[4a^2(1 - \mu_L\lambda) - b^2\mu_H\lambda]}\end{aligned}$$

where λ fully characterizes σ . ■

Proof. Proposition 3. This proof applies to a more general result which states that is optimal for the designer to select $\sigma(p_2|\theta)$ to be degenerated for any θ . Here I present the proof for $\sigma(p_2|\theta_H)$ but the proof for the other case is analogous.

Suppose not. Assume that the optimal recommendation mechanism $\sigma^* = \{\sigma^*(p_2|\theta)\}_{\theta \in \Theta}$ is partially informative where both $\sigma^*(p_2|\theta)$ are not degenerated. Consider an alternative partially informative recommendation mechanism $\hat{\sigma}$ in which $\hat{\sigma}(p_2|\theta_H)$ is degenerated and places all its mass on one point $\hat{p}_H \in [p^F(\theta_L), p^F(\theta_H)] = [p_L^F, p_H^F]$ where $\hat{p}_H \in \text{supp } \hat{\sigma}(p_2|\theta_L)$. By Lemma 15, for any obedient $\hat{\sigma}$, the support of $\hat{\sigma}|\theta_L$ is $\{\hat{p}_L, \hat{p}_H\}$ where \hat{p}_L and \hat{p}_H are defined in Lemma 14 and $\lambda = \hat{\sigma}(\hat{p}_L|\theta_L)$ fully characterizes $\hat{\sigma}$. Next, I show that there exists $\lambda \in (0, 1)$ such that $\Delta\mathbb{E}[CS](\hat{\sigma}) \geq \Delta\mathbb{E}[CS](\sigma^*)$. Given that $\mathbb{E}_\sigma[p_2] = \mathbb{E}_{\sigma'}[p_2]$ for all feasible σ, σ' ,²⁶ the difference between $\Delta\mathbb{E}[CS](\hat{\sigma})$ and $\Delta\mathbb{E}[CS](\sigma^*)$, denoted as $\Delta\mathbb{E}[CS]_{\hat{\sigma}-\sigma^*}$, is

$$\begin{aligned}\Delta\mathbb{E}[CS]_{\hat{\sigma}-\sigma^*} &= \frac{a}{2} (1 + \delta^2) (\mathbb{E}_{\hat{\sigma}}[p_2^2] - \mathbb{E}_{\sigma^*}[p_2^2]) - \left[\left(1 - \frac{\delta^2}{2}\right) \left(1 + \frac{\delta}{2}\right) - \frac{\delta}{4} \right] (\mathbb{E}_{\hat{\sigma}}[\theta \cdot p_2] - \mathbb{E}_{\sigma^*}[\theta \cdot p_2]) \\ &\quad - \frac{b\delta}{8} (7 - \delta^2) (\mathbb{E}_{\hat{\sigma}}[\mathbb{E}[p_2|\theta]^2] - \mathbb{E}_{\sigma^*}[\mathbb{E}[p_2|\theta]^2])\end{aligned}$$

For any feasible σ^* , the expectation $\mathbb{E}_{\sigma^*}[p_2|\theta_L]$ satisfies

$$\mathbb{E}_{\sigma^*}[p_2|\theta_L] \in \left(\frac{\theta_L}{2a - b}, \frac{\mathbb{E}_\mu[\theta]}{2a - b} \right).$$

²⁶Note that $\mathbb{E}_\pi[p_2] = \mathbb{E}_{\pi'}[p_2]$ for all feasible π, π' since

$$\mathbb{E}_\pi[p_2] = \frac{1}{2a} \left[\mathbb{E}[\theta] \left(1 + \frac{b}{2a}\right) + \frac{b^2}{2a} \mathbb{E}_\pi[p_2] \right] \Leftrightarrow \mathbb{E}_\pi[p_2] = \frac{\mathbb{E}_\mu[\theta]}{2a - b}.$$

The equality holds by the uninformed firm's and informed firm's best response functions and by the law of iterated expectations. Then, $\mathbb{E}_\pi[p_2]$ doesn't depend on π . Given the equivalence between π_2 and σ , it also follows that $\mathbb{E}_\sigma[p_2] = \mathbb{E}_{\sigma'}[p_2]$ for all feasible σ, σ' .

Moreover, by definition, $\mathbb{E}_{\hat{\sigma}}[p_2|\theta_L] = \lambda\hat{p}_L + (1 - \lambda)\hat{p}_H$, and

$$\mathbb{E}_{\hat{\sigma}}[p_2|\theta_L] = \frac{\theta_L}{2a - b} \text{ if } \lambda = 1 \text{ and } \mathbb{E}_{\hat{\sigma}}[p_2|\theta_L] = \frac{\mathbb{E}[\theta]}{2a - b} \text{ if } \lambda = 0.$$

The intermediate value theorem implies that there exists $\tilde{\lambda} \in (0, 1)$ such that $\mathbb{E}_{\hat{\sigma}}[p_2|\theta_L] = \mathbb{E}_{\sigma^*}[p_2|\theta_L]$ since $\mathbb{E}_{\hat{\sigma}}[p_2|\theta_L]$ is a continuous function of λ . Since $\mathbb{E}_{\sigma}[p_2] = \mathbb{E}_{\sigma'}[p_2]$ for all feasible σ and σ' , $\tilde{\lambda}$ also satisfies $\mathbb{E}_{\hat{\sigma}}[p_2|\theta_H] = \mathbb{E}_{\sigma^*}[p_2|\theta_H]$. Then, the difference between $\Delta\mathbb{E}[CS](\hat{\sigma})$ and $\Delta\mathbb{E}[CS](\sigma^*)$ for $\hat{\sigma}$ characterized by $\tilde{\lambda}$ is

$$\begin{aligned} \Delta\mathbb{E}[CS]_{\hat{\sigma}-\sigma^*} &= \frac{a}{2} (1 + \delta^2) [\mathbb{E}_{\hat{\sigma}}[p_2^2] - \mathbb{E}_{\sigma^*}[p_2^2]] \\ &= \frac{a}{2} (1 + \delta^2) \left[\mu_L \left(\tilde{\lambda}\hat{p}_L^2 + (1 - \tilde{\lambda})\hat{p}_H^2 - \mathbb{E}_{\sigma^*}[p_2^2|\theta_L] \right) + \mu_H \left(\hat{p}_H^2 - \mathbb{E}_{\sigma^*}[p_2^2|\theta_H] \right) \right] \end{aligned}$$

Hence, $\mathbb{E}_{\hat{\sigma}}[p_2^2] \geq \mathbb{E}_{\sigma^*}[p_2^2]$ by Jensen's inequality. Then, for all demand parameters and σ^* such that $\delta \leq \hat{c}$, there exists $\lambda \in (0, 1)$ such that $\Delta\mathbb{E}[CS]_{\hat{\sigma}-\sigma^*} \geq 0$. This contradicts the optimality of σ^* . Thus, the optimal partially informative recommendation mechanism is such that $\text{supp } \sigma|\theta_H = \{\hat{p}_H\}$ and $\text{supp } \sigma|\theta_L = \{\hat{p}_L, \hat{p}_H\}$.

Lastly, the optimal recommendation mechanism is characterized by $\lambda^* \in \arg \max_{\lambda \in [0, 1]} \Delta\mathbb{E}[CS](\lambda)$ where

$$\begin{aligned} \Delta\mathbb{E}[CS](\lambda) &= \frac{a}{2} (\delta^2 + 1) \mu_L \lambda [\mu_L(1 - \lambda) + \mu_H] (\hat{p}_H - \hat{p}_L)^2 \\ &\quad - \left[\left(1 - \frac{\delta^2}{2} \right) \left(\frac{\delta}{2} + 1 \right) - \frac{\delta}{4} \right] \left[\mu_L \theta_L [\lambda\hat{p}_L + (1 - \lambda)\hat{p}_H] + \mu_H \theta_H \hat{p}_H - \frac{(\mu_L \theta_L + \mu_H \theta_H)^2}{2a - b} \right] \\ &\quad - \frac{b\delta}{8} (7 - \delta^2) \left[\mu_L [\lambda\hat{p}_L + (1 - \lambda)\hat{p}_H]^2 + \mu_H \hat{p}_H^2 - [\mu_L \lambda\hat{p}_L + (1 - \mu_L \lambda)\hat{p}_H]^2 \right] \end{aligned}$$

and \hat{p}_L and \hat{p}_H are functions of λ defined in Lemma 14. The optimal $\lambda^* \in (0, 1)$ is characterized by the first order condition of $\Delta\mathbb{E}[CS](\lambda)$ and it is given by

$$\lambda^* = \frac{4 [\delta(1 - 3\delta^2) + 6(1 - \delta^2)]}{\mu_H \delta^5 + 2\mu_H \delta^4 - (12 - \mu_H)\delta^3 - 6(4 - \mu_H)\delta^2 + 4(1 - \mu_H)\delta + 24(1 - \mu_H)}.$$

■

A.2.4 Producer surplus optimal disclosure: proofs

Proof. Lemma 5. For the informed firm, the difference in expected profits with full disclosure σ^F and any disclosure σ is

$$\Pi_2(\sigma^F) - \Pi_2(\sigma) = a\mathbb{E}_{\mu} \left[\left(\frac{\theta}{2a - b} \right)^2 - 2\frac{\theta}{2a - b} \mathbb{E}_{\sigma}[p_2|\theta] + \mathbb{E}_{\sigma}[p_2^2|\theta] \right] + \frac{b^2}{2a} \mathbb{E}_{\mu} \left[\frac{\theta}{2a - b} \mathbb{E}_{\sigma}[p_2|\theta] - \mathbb{E}_{\sigma}[p_2|\theta]^2 \right]$$

Then, this difference is positive since

$$\begin{aligned}\Pi_2(\sigma^F) - \Pi_2(\sigma) &\geq a\mathbb{E}_\mu \left[\left(\frac{\theta}{2a-b} - \mathbb{E}_\sigma[p_2|\theta] \right)^2 \right] + \frac{b^2}{2a} \sum_{\theta \in \Theta} \mu_\theta \left[\mathbb{E}_\sigma[p_2|\theta] \left(\frac{\theta}{2a-b} - \mathbb{E}_\sigma[p_2|\theta] \right) \right] \\ &\geq a\mathbb{E}_\mu \left[\left(\frac{\theta}{2a-b} - \mathbb{E}_\sigma[p_2|\theta] \right)^2 \right] \geq 0\end{aligned}$$

The first inequality holds by Jensen's inequality. The second since $a > |b| > 0$,

$$\frac{\theta_L}{2a-b} \leq \mathbb{E}_\sigma[p_2|\theta_L] \leq \mathbb{E}_\sigma[p_2|\theta_H] \leq \frac{\theta_H}{2a-b} \text{ and } \sum_{\theta \in \Theta} \mu_\theta \mathbb{E}_\sigma[p_2|\theta] = \frac{\mathbb{E}_\mu[\theta]}{2a-b}$$

for all feasible σ . Hence, $\Pi_2(\sigma^F) \geq \Pi_2(\sigma)$ which implies that full disclosure is optimal for the uninformed firm. ■

A.2.5 Welfare optimal disclosure: proofs

Proof. Proposition 4. First, I show that full disclosure yields a higher expected welfare than no disclosure if $\delta \geq \tilde{\alpha}$. First, full disclosure σ^F yields a higher expected welfare than no disclosure σ^N if and only if $\delta \geq \tilde{\alpha}$. Hence, if $\delta \geq \tilde{\alpha}$, either full or partial disclosure is optimal whereas if $\delta < \tilde{\alpha}$, either no or partial disclosure is optimal.

Second, consider $\delta < \tilde{\alpha}$. The difference between the total expected surplus with partial disclosure σ and no disclosure σ^N is given by

$$\begin{aligned}\mathbb{E}_{(\mu,\sigma)} [W((p_1, p_2); \theta)] - \mathbb{E}_{(\mu,\sigma^N)} [W((p_1, p_2); \theta)] \\ = \frac{\delta}{2} \left[\delta \left(\frac{\delta}{2} + 1 \right) + \frac{3}{2} \right] Cov_{(\mu,\sigma)}[\theta, p_2] - \frac{a}{2} (1 - \delta^2) \mathbb{V}_{(\mu,\sigma)}[p_2] - \frac{b}{8} \delta (1 - \delta^2) \mathbb{V}_\mu[\mathbb{E}_\sigma[p_2|\theta]]\end{aligned}$$

This difference is a continuous and strictly increasing function of δ . Moreover, as δ converges to 0, the difference in expected consumer surplus converges to

$$\mathbb{E}_{(\mu,\sigma)} [W((p_1, p_2); \theta)] - \mathbb{E}_{(\mu,\sigma^N)} [W((p_1, p_2); \theta)] \xrightarrow{\delta \rightarrow 0} -\frac{a}{2} \mathbb{V}_{(\mu,\sigma)}[p_2] < 0$$

and Lemma 12 shows that $\mathbb{E}_{(\mu,\sigma)} [W((p_1, p_2); \theta)] > \mathbb{E}_{(\mu,\sigma^N)} [W((p_1, p_2); \theta)]$ for all $\delta > \tilde{\alpha}_1$. Then, the intermediate value theorem implies that there exists a $\tilde{\alpha}_1 \in (0, \tilde{\alpha}]$ such that

$$\mathbb{E}_{(\mu,\sigma)} [W((p_1, p_2); \theta)] = \mathbb{E}_{(\mu,\sigma^N)} [W((p_1, p_2); \theta)].$$

Also, since this difference is strictly increasing in δ , this also implies that

$$\mathbb{E}_{(\mu,\sigma)} [W((p_1, p_2); \theta)] > \mathbb{E}_{(\mu,\sigma^N)} [W((p_1, p_2); \theta)] \text{ for all } \delta > \tilde{\alpha}_1 \text{ and}$$

$$\mathbb{E}_{(\mu,\sigma)} [W((p_1, p_2); \theta)] < \mathbb{E}_{(\mu,\sigma^N)} [W((p_1, p_2); \theta)] \text{ for all } \delta < \tilde{\alpha}_1.$$

That is, partial disclosure is welfare if $\delta \in [\tilde{\alpha}_1, \tilde{\alpha})$ and no disclosure is welfare optimal if $\delta < \tilde{\alpha}_1$.

Now, consider $\delta \geq \tilde{\alpha}$. The difference between expected welfare with full disclosure $\mathbb{E}_{(\mu,\sigma^F)} [W((p_1, p_2); \theta)]$ and with partial disclosure $\mathbb{E}_{(\mu,\sigma)} [W((p_1, p_2); \theta)]$ is given by

$$\begin{aligned} & \mathbb{E}_{(\mu,\sigma^F)} [W((p_1, p_2); \theta)] - \mathbb{E}_{(\mu,\sigma)} [W((p_1, p_2); \theta)] \\ &= \frac{b\delta}{2} \left(\delta^2 - \frac{3}{2} \right) \mathbb{E}_\mu \left[\left(\frac{\theta}{2a-b} \right)^2 - \mathbb{E}_\sigma [p_2|\theta]^2 \right] - \frac{b\delta}{2} \mathbb{E}_\mu \left[\mathbb{E}_\sigma [p_2^2|\theta] - \mathbb{E}_\sigma [p_2|\theta]^2 \right] \\ &+ \delta \left(\frac{3}{4} + \frac{\delta}{2} + \frac{\delta^2}{4} \right) \mathbb{E}_\mu \left[\theta \left(\frac{\theta}{2a-b} - \mathbb{E}_\sigma [p_2|\theta] \right) \right] - \frac{a}{2} \mathbb{E}_\mu \left[\left(\frac{\theta}{2a-b} \right)^2 - \mathbb{E}_\sigma [p_2^2|\theta] \right], \end{aligned}$$

which is a continuous function of δ .

First, as $\delta \rightarrow 1$, this difference converges to

$$\mathbb{E}_{(\mu,\sigma^F)} [W((p_1, p_2); \theta)] - \mathbb{E}_{(\mu,\sigma)} [W((p_1, p_2); \theta)] \xrightarrow{\delta \rightarrow 1} \frac{3}{4a} \mathbb{E}_\mu \left[\left(\frac{\theta}{a} - \mathbb{E}_\sigma [p_2|\theta] \right)^2 \right] > 0$$

Second, Lemma 12 shows that partial disclosure yields higher expected welfare than full disclosure if $\delta < \tilde{c}_2$. Then, the intermediate value theorem implies that there exists $\tilde{\alpha}_2 \in [\tilde{c}_2, 1)$ such that

$$\mathbb{E}_{(\mu,\sigma^F)} [W((p_1, p_2); \theta)] = \mathbb{E}_{(\mu,\sigma)} [W((p_1, p_2); \theta)].$$

Analogously as before, $\mathbb{E}_{(\mu,\sigma^F)} [W((p_1, p_2); \theta)] - \mathbb{E}_{(\mu,\sigma)} [W((p_1, p_2); \theta)] > 0$ for $\delta > \tilde{\alpha}_2$ and negative for $\delta \in [\alpha, \tilde{\alpha}_2)$. Hence, full disclosure is welfare optimal if $\delta > \tilde{\alpha}_2$ and partial disclosure is optimal if $\delta \in [\alpha, \tilde{\alpha}_2)$. In summary, full disclosure is welfare optimal if $\delta \geq \tilde{\alpha}_2$, partial disclosure is welfare optimal if $\delta \in (\tilde{\alpha}_1, \tilde{\alpha}_2)$ and no disclosure is welfare optimal if $\delta \in (0, \tilde{\alpha}_1]$. ■

Proof. Proposition 5. The proof is analogous to the proof of Proposition 3. Suppose not. Assume that the optimal recommendation mechanism $\sigma^* = \{\sigma^*(p_2|\theta)\}_{\theta \in \Theta}$ is partially informative where both $\sigma^*(p_2|\theta)$ are not degenerated. Consider an alternative partially informative recommendation mechanism $\hat{\sigma}$ in which $\hat{\sigma}(p_2|\theta_H)$ is degenerated and places all its mass on one point $\hat{p}_H \in [p^F(\theta_L), p^F(\theta_H)] = [p_L^F, p_H^F]$ where $\hat{p}_H \in \text{supp } \hat{\sigma}(p_2|\theta_L)$. By Lemma 15, for any obedient $\hat{\sigma}$, the support of $\hat{\sigma}|\theta_L$ is $\{\hat{p}_L, \hat{p}_H\}$ where \hat{p}_L and \hat{p}_H are defined in Lemma 14 and $\lambda = \hat{\sigma}(\hat{p}_L|\theta_L)$ fully characterizes $\hat{\sigma}$.

Consider first the case in which $\delta < \tilde{\alpha}$. Next, I show that there exists $\lambda \in (0, 1)$ such that $\mathbb{E}[W](\hat{\sigma}) - \mathbb{E}[W](\sigma^N) \geq \mathbb{E}[W](\sigma^*) - \mathbb{E}[W](\sigma^N)$ where

$$\mathbb{E}[W](\sigma) - \mathbb{E}[W](\sigma^N) = \frac{\delta}{2} \left[\delta \left(\frac{\delta}{2} + 1 \right) + \frac{3}{2} \right] Cov_{(\mu, \sigma)}[\theta, p_2] - \frac{a}{2} (1 - \delta^2) \mathbb{V}_{(\mu, \sigma)}[p_2] - \frac{b}{8} \delta (1 - \delta^2) \mathbb{V}_\mu[\mathbb{E}_\sigma[p_2|\theta]]$$

The difference between $\mathbb{E}[W](\hat{\sigma}) - \mathbb{E}[W](\sigma^N)$ and $\mathbb{E}[W](\sigma^*) - \mathbb{E}[W](\sigma^N)$ is

$$\Delta \mathbb{E}[W]_{\hat{\sigma}-\sigma^*}^N \geq \frac{\delta}{2} \left[\delta \left(\frac{\delta}{2} + 1 \right) + \frac{3}{2} \right] (\mathbb{E}_{\hat{\sigma}}[\theta \cdot p_2] - \mathbb{E}_{\sigma^*}[\theta \cdot p_2]) - \left(\frac{a}{2} + \frac{b}{8} \delta \right) (1 - \delta^2) (\mathbb{E}_{\hat{\sigma}}[p_2^2] - \mathbb{E}_{\sigma^*}[\mathbb{E}[p_2|\theta]^2])$$

where the inequality holds since $\mathbb{E}_\sigma[p_2] = \mathbb{E}_{\sigma'}[p_2]$ for all feasible σ, σ' and $\mathbb{E}_\sigma[p_2^2] \geq \mathbb{E}_\sigma[\mathbb{E}[p_2|\theta]^2]$ for all σ . Note that $\mathbb{E}_{\hat{\sigma}}[p_2^2]$ is a continuous function of λ and for any σ^* ,

$$\begin{aligned} \mathbb{E}_{\sigma^*}[\mathbb{E}[p_2|\theta]^2] &\in \left(\left(\frac{\mathbb{E}[\theta]}{2a-b} \right)^2, \mathbb{E}_{\sigma^*}[p_2^2] \right) \subset \left(\left(\frac{\mathbb{E}[\theta]}{2a-b} \right)^2, \mu_L \left(\frac{\theta_L}{2a-b} \right)^2 + \mu_H \left(\frac{\theta_H}{2a-b} \right)^2 \right) \\ \mathbb{E}_{\hat{\sigma}}[p_2^2] &= \left(\frac{\mathbb{E}[\theta]}{2a-b} \right)^2 \text{ if } \lambda = 0 \text{ and } \mathbb{E}_{\hat{\sigma}}[p_2^2] = \mu_L \left(\frac{\theta_L}{2a-b} \right)^2 + \mu_H \left(\frac{\theta_H}{2a-b} \right)^2 \text{ if } \lambda = 1 \end{aligned}$$

Hence, the intermediate value theorem implies that there exists $\hat{\lambda} \in (0, 1)$ such that $\mathbb{E}_{\hat{\sigma}}[p_2^2] = \mathbb{E}_{\sigma^*}[\mathbb{E}[p_2|\theta]^2]$. Then, the difference between $\mathbb{E}[W](\hat{\sigma}) - \mathbb{E}[W](\sigma^N)$ and $\mathbb{E}[W](\sigma^*) - \mathbb{E}[W](\sigma^N)$ for $\hat{\lambda}$ satisfies

$$\begin{aligned} \Delta \mathbb{E}[W]_{\hat{\sigma}-\sigma^*}^N &\geq \frac{\delta}{2} \left[\delta \left(\frac{\delta}{2} + 1 \right) + \frac{3}{2} \right] (\mathbb{E}_{\hat{\sigma}}[\theta \cdot p_2] - \mathbb{E}_{\sigma^*}[\theta \cdot p_2]) \\ &\geq \frac{\delta}{2} \left[\delta \left(\frac{\delta}{2} + 1 \right) + \frac{3}{2} \right] (\theta_L \mathbf{1}_{(\mathbb{E}_{\hat{\sigma}}[p_2|\theta] \geq \mathbb{E}_{\sigma^*}[p_2|\theta])} + \theta_H \mathbf{1}_{(\mathbb{E}_{\hat{\sigma}}[p_2|\theta] < \mathbb{E}_{\sigma^*}[p_2|\theta])}) [\mathbb{E}_{\hat{\sigma}}[p_2] - \mathbb{E}_{\sigma^*}[p_2]] \end{aligned}$$

where $\mathbb{E}_{\hat{\sigma}}[p_2] = \mathbb{E}_{\sigma^*}[p_2]$, contradicting the optimality of σ^* . Thus, the optimal disclosure has binary support and it is characterized by $\lambda^* \in \arg \max_{\lambda \in [0, 1]} \Delta \mathbb{E}[W]^N(\lambda)$ where

$$\begin{aligned} \Delta \mathbb{E}[W]^N(\lambda) &= \frac{\delta}{2} \left[\delta \left(\frac{\delta}{2} + 1 \right) + \frac{3}{2} \right] \left[\mu_L \theta_L [\lambda \hat{p}_L + (1 - \lambda) \hat{p}_H] + \mu_H \theta_H \hat{p}_H - \frac{(\mu_L \theta_L + \mu_H \theta_H)^2}{2a-b} \right] \\ &\quad - \frac{a}{2} (1 - \delta^2) \mu_L \lambda [\mu_L (1 - \lambda) + \mu_H] (\hat{p}_H - \hat{p}_L)^2 \\ &\quad - \frac{b}{8} \delta (1 - \delta^2) \left[\mu_L [\lambda \hat{p}_L + (1 - \lambda) \hat{p}_H]^2 + \mu_H \hat{p}_H^2 - [\mu_L \lambda \hat{p}_L + (1 - \mu_L \lambda) \hat{p}_H]^2 \right] \end{aligned}$$

The optimal $\lambda^* \in (0, 1)$ is characterized by the first order condition of $\Delta \mathbb{E}[W]^N(\lambda)$.

Consider now the case in which $\delta \geq \tilde{\alpha}$. Next, I show that

$$\mathbb{E}[W](\hat{\sigma}) - \mathbb{E}[W](\sigma^F) \geq \mathbb{E}[W](\sigma^*) - \mathbb{E}[W](\sigma^F).$$

The difference between $\mathbb{E}[W](\hat{\sigma}) - \mathbb{E}[W](\sigma^F)$ and $\mathbb{E}[W](\sigma^*) - \mathbb{E}[W](\sigma^F)$, denoted by $\Delta\mathbb{E}[W]_{\hat{\sigma}-\sigma^*}^F$, is

$$\begin{aligned}\Delta\mathbb{E}[W]_{\hat{\sigma}-\sigma^*}^F &= \frac{\delta}{2} \left[\left(\frac{\delta}{2} + 1 \right) \delta + \frac{3}{2} \right] (\mathbb{E}_{\hat{\sigma}}[\theta \cdot p_2] - \mathbb{E}_{\sigma^*}[\theta \cdot p_2]) - \frac{a}{2} (1 - \delta^2) (\mathbb{E}_{\hat{\sigma}}[p_2^2] - \mathbb{E}_{\sigma^*}[p_2^2]) \\ &\quad - \frac{b\delta}{2} \left(\frac{5}{2} - \delta^2 \right) (\mathbb{E}_{\hat{\sigma}}[\mathbb{E}[p_2|\theta]^2] - \mathbb{E}_{\sigma^*}[\mathbb{E}[p_2|\theta]^2])\end{aligned}$$

Analogously as before, there exists $\hat{\lambda} \in (0, 1)$ such that $\mathbb{E}_{\hat{\sigma}}[p_2^2] = \mathbb{E}_{\sigma^*}[\mathbb{E}[p_2|\theta]^2]$ and the difference between $\mathbb{E}[W](\hat{\sigma}) - \mathbb{E}[W](\sigma^F)$ and $\mathbb{E}[W](\sigma^*) - \mathbb{E}[W](\sigma^F)$ for $\hat{\lambda}$ satisfies

$$\Delta\mathbb{E}[W]_{\hat{\sigma}-\sigma^*}^F \geq \frac{\delta}{2} \left[\left(\frac{\delta}{2} + 1 \right) \delta + \frac{3}{2} \right] (\mathbb{E}_{\hat{\sigma}}[\theta \cdot p_2] - \mathbb{E}_{\sigma^*}[\theta \cdot p_2])$$

and since $\mathbb{E}_{\hat{\sigma}}[\theta \cdot p_2] \geq \mathbb{E}_{\sigma^*}[\theta \cdot p_2]$, this contradicts the optimality of σ^* . Thus, the optimal partially informative recommendation mechanism is such that $\text{supp } \sigma|\theta_H = \{\hat{p}_H\}$ and $\text{supp } \sigma|\theta_L = \{\hat{p}_L, \hat{p}_H\}$. The optimal disclosure is characterized by $\lambda^* \in \arg \max_{\lambda \in [0,1]} \Delta\mathbb{E}[W]^F(\lambda)$ where

$$\begin{aligned}\Delta\mathbb{E}[W]^F(\lambda) &= \frac{\delta}{2} \left[\delta \left(\frac{\delta}{2} + 1 \right) + \frac{3}{2} \right] \left[\mu_L \theta_L [\lambda \hat{p}_L + (1 - \lambda) \hat{p}_H] + \mu_H \theta_H \hat{p}_H - \frac{(\mu_L \theta_L + \mu_H \theta_H)^2}{2a - b} \right] \\ &\quad - \frac{a}{2} (1 - \delta^2) \mu_L \lambda [\mu_L (1 - \lambda) + \mu_H] (\hat{p}_H - \hat{p}_L)^2 \\ &\quad - \frac{b}{2} \delta \left(\frac{5}{2} - \delta^2 \right) \left[\mu_L [\lambda \hat{p}_L + (1 - \lambda) \hat{p}_H]^2 + \mu_H \hat{p}_H^2 - [\mu_L \lambda \hat{p}_L + (1 - \mu_L \lambda) \hat{p}_H]^2 \right]\end{aligned}$$

The optimal $\lambda^* \in (0, 1)$ is characterized by the first order condition of $\Delta\mathbb{E}[W]^F(\lambda)$. ■

A.3 Non-linear demand

Consider the same environment as before but assume firm i 's demand, $q(p_i, p_{-i}; \theta)$, is continuous and differentiable and satisfies the following properties:

$$i) \frac{\partial q(p_i, p_{-i}; \theta)}{\partial p_i} \leq 0, \quad ii) \quad \frac{\partial q(p_i, p_{-i}; \theta)}{\partial \theta} > 0, \quad \text{and} \quad iii) \quad \left| \frac{\partial q(p_i, p_{-i}; \theta)}{\partial p_i} \right| > \left| \frac{\partial q(p_i, p_{-i}; \theta)}{\partial p_{-i}} \right|$$

The first condition ensures that quantity demanded decreases as price increases, the second condition implies that the state is a positive demand shifter and, lastly, the third condition implies that goods are differentiated and that a change of its own price has a bigger effect on the demand than a change of the price of a competitor.²⁷ Assume that firm's ex-post profits are strictly concave in p_i . That is,

$$p_i \frac{\partial^2 q(p_i, p_{-i}; \theta)}{\partial p_i^2} < -2 \frac{\partial q(p_i, p_{-i}; \theta)}{\partial p_i} \quad \text{for all } p_i.$$

²⁷This ensures that equilibrium prices are finite.

Furthermore, assume that

$$\frac{\partial^2 \Pi_i(p_i, p_{-i}; \theta)}{\partial p_i^2} \frac{\partial^2 \Pi_{-i}(p_{-i}, p_i; \theta)}{\partial p_{-i}^2} \geq \left(\frac{\partial^2 \Pi_i(p_i, p_{-i}; \theta)}{\partial p_i \partial p_{-i}} \right)^2.$$

Firms offer substitutes (complements) if

$$\frac{\partial q(p_i, p_{-i}; \theta)}{\partial p_{-i}} > 0 (< 0).$$

When firms offer substitutes, assume that the elasticity of demand of firm i is a non-increasing function of the other firm's price and that the demand is supermodular in the state θ and the price of the other firm p_{-i} , i.e.,

$$\frac{\partial^2 q(p_i, p_{-i}; \theta)}{\partial p_i \partial p_{-i}} \geq 0 \text{ for all } (p_i, p_{-i}) \text{ and } \frac{\partial^2 q(p_i, p_{-i}; \theta)}{\partial \theta \partial p_{-i}} \geq 0.$$

Similarly, when firms offer complements, assume that the elasticity of demand of firm i is a non-decreasing function of the other firm's price and that the demand is submodular in the state θ and the price of the other firm. That is,

$$\frac{\partial^2 q(p_i, p_{-i}; \theta)}{\partial p_i \partial p_{-i}} \leq 0 \text{ for all } (p_i, p_{-i}) \text{ and } \frac{\partial^2 q(p_i, p_{-i}; \theta)}{\partial \theta \partial p_{-i}} \leq 0.$$

Note that these assumptions imply that firms' choices are strategic complements (substitutes) when they offer substitutes (complements).

Pricing game equilibrium. For all information structures (S_2, π_2) , the existence and uniqueness of the BNE is guaranteed by Ui (2016), which provides sufficient conditions for the existence and uniqueness of BNE in Bayesian games with concave and continuously differentiable payoff functions. This is formalized in Lemma 16.

Lemma 16 *For all information structures (S_2, π_2) , the set of Bayesian Nash equilibria in the pricing game $\hat{\mathcal{E}}(S_2, \pi_2)$ is a singleton.*

Simplifications. The strict concavity of firm's ex-post profits in p_i imply that firms' profits are bounded and continuous functions and that there exists \bar{p} such that it is without loss of generality to restrict attention to the compact action space $p_i \in [0, \bar{p}]$. The equivalence to recommendation mechanism σ is established in Lemma 2 and Lemma 3. The existence and uniqueness of BNE imply that it is sufficient to restrict attention to the distribution $\sigma(p_2|\theta)$ since for any obedient recommendation mechanism there exists a function $p_1(\theta, \sigma(p_2|\theta))$

which represents firm 1's best response when the state is θ and the price recommendations are given by σ where

$$p_1(\theta, \sigma(p_2|\theta)) = \arg \max_{p_1} \int_{p_2} v_1(p_1, p_2; \theta) d\sigma(p_2|\theta).$$

By Leibniz rule, $p_1(\theta, \sigma(p_2|\theta))$ is implicitly characterized by

$$\int_{p_2} q(p_1, p_2; \theta) d\sigma(p_2|\theta) + p_1 \int_{p_2} \frac{\partial q(p_1, p_2; \theta)}{\partial p_1} d\sigma(p_2|\theta) = 0.$$

Then, firm 1's expected equilibrium profits given information structure σ are

$$\mathbb{E}_{(\mu, \sigma)}[\Pi_1^*(p_1, p_2; \theta)] = \sum_{\theta \in \Theta} \mu_\theta \mathbb{E}_\sigma[\Pi_1^*(p_1, p_2; \theta)|\theta] = \sum_{\theta \in \Theta} \mu_\theta \int_{p_2} \Pi_1(p_1(\theta, \sigma(p_2|\theta)), p_2; \theta) d\sigma(p_2|\theta)$$

Furthermore, for any information structure σ , the set of recommended equilibrium prices for firm 2 in the pricing game is a subset of the interval between the equilibrium prices with full disclosure. This is formalized in Lemma 17.

Lemma 17 *The support of any obedient distribution $\sigma(p_2|\theta)$ is a subset of $[p^F(\theta_L), p^F(\theta_H)]$ for all $\theta \in \Theta$ where $p^F(\theta)$ is the equilibrium price with full disclosure when the state θ is realized.*

Informed firm optimal disclosure. Assume the designer wants to maximize the informed firm's expected equilibrium payoffs. First, I show that when firms offer substitutes, the informed firm's expected equilibrium payoff conditional on the state is supermodular in the state and the price of the other firm. Similarly, I also show that the informed firm's expected equilibrium payoff conditional on the state is submodular in the state and the price of the other firm when firms offer complements. This is formalized in Lemma 18.

Lemma 18 *When firms offer substitutes (complements), $\mathbb{E}_\sigma[\Pi_1^*(p_1, p_2; \theta)|\theta]$ is supermodular (submodular) in θ and p_2 .*

Second, I show that it is optimal for the informed firm to share all its private information with the uninformed firm when the informed firm expected equilibrium profits are supermodular in the state and the uninformed firm's price. I also show that it is optimal for the informed firm to share none of its private information with the uninformed firm when the informed firm expected equilibrium profits are submodular in the state and the uninformed firm's price. This is formalized in Proposition 6.

Proposition 6 *If $\mathbb{E}_\sigma[\Pi_1^*(p_1, p_2; \theta)|\theta]$ is supermodular in p_2 and θ , full disclosure is optimal for the informed firm. Similarly, if $\mathbb{E}_\sigma[\Pi_1^*(p_1, p_2; \theta)|\theta]$ is submodular in p_2 and θ , no disclosure is optimal for the informed firm.*

These two results imply that full disclosure is optimal for the informed firm when firms offer substitutes and no disclosure is optimal when firms offer complements. These results also extend to Cournot competition using same equivalence arguments as before.

A.3.1 Proofs

Proof. Lemma 16. The pricing game is a smooth concave game since $\Pi_i(\cdot, p_{-i}; \theta) : \mathbb{R}_+ \rightarrow \mathbb{R}$ is concave and continuously differentiable for each $p_{-i} \in \mathbb{R}_+$ since

$$\frac{\partial^2 \Pi_i(p_i, p_{-i}; \theta)}{\partial p_i^2} < 0 \text{ for all } p_{-i} \in \mathbb{R}_+.$$

Define the payoff gradient as

$$\nabla \Pi(\mathbf{p}, \theta) := \left(\frac{\partial \Pi_i((p_i, p_{-i}); \theta)}{\partial p_i} \right)_{i \in \{1, 2\}}.$$

The payoff gradient is continuously differentiable. The Jacobian matrix of the payoff gradient, given by

$$F_{\nabla \Pi}(\mathbf{p}, \theta) := \begin{pmatrix} \frac{\partial^2 \Pi_1((p_1, p_2); \theta)}{\partial p_1^2} & \frac{\partial^2 \Pi_1((p_1, p_2); \theta)}{\partial p_1 \partial p_2} \\ \frac{\partial^2 \Pi_2((p_2, p_1); \theta)}{\partial p_1 \partial p_2} & \frac{\partial^2 \Pi_2((p_2, p_1); \theta)}{\partial p_2^2} \end{pmatrix},$$

is negative definite because

$$\frac{\partial^2 \Pi_i(p_i, p_{-i}; \theta)}{\partial p_i^2} < 0 \text{ and } \frac{\partial^2 \Pi_i(p_i, p_{-i}; \theta)}{\partial p_i^2} \frac{\partial^2 \Pi_{-i}(p_{-i}, p_i; \theta)}{\partial p_{-i}^2} \geq \left(\frac{\partial^2 \Pi_i(p_i, p_{-i}; \theta)}{\partial p_i \partial p_{-i}} \right)^2.$$

This implies that the payoff gradient $\nabla \Pi(\mathbf{p}, \theta)$ is strictly monotone by Lemma 4 from Ui (2016). Furthermore, since for all $\mathbf{p} := (p_i, p_{-i})$, there exists $c > 0$ such that

$$\mathbf{p}^T F_{\nabla \Pi}(\mathbf{p}, \theta) \mathbf{p} < -c \mathbf{p}^T \mathbf{p},$$

the payoff gradient is also strongly monotone by the same lemma. Then, the uniqueness of the Bayesian Nash equilibrium of the pricing game follows from Proposition 1 from Ui (2016), which states that if the payoff gradient is strictly monotone, the Bayesian game as at most one Bayesian Nash equilibrium. The existence of a unique Bayesian Nash equilibrium follows from Proposition 2 from Ui (2016). ■

Proof. Lemma 17. With full disclosure, there is no uncertainty about the state. Each firm chooses $p_i : \Theta \rightarrow \mathbb{R}_+$ to maximize $\Pi_i(p_i, p_{-i}; \theta)$. That is, firm i 's best response to p_{-i} is implicitly defined by

$$q(p_i, p_{-i}; \theta) + p_i \frac{\partial q(p_i, p_{-i}; \theta)}{\partial p_i} = 0$$

In equilibrium, both firms choose the same price, denoted by $p^F(\theta)$. Since $q(p_2, p_1; \theta_L) < q(p_2, p_1, \theta_H)$, the highest (lowest) equilibrium price the uninformed firm is willing to price is when both firms are certain that the state is high (low). Hence, the support of any obedient recommendation $\sigma(p_2|\theta)$ is a subset of $[p^F(\theta_L), p^F(\theta_H)]$. ■

Proof. Lemma 18. By definition, $\mathbb{E}_\sigma[\Pi_1^*(p_1, p_2; \theta)|\theta]$ is given by

$$\mathbb{E}_\sigma[\Pi_1^*(p_1, p_2; \theta)|\theta] = \int_{p_2} \Pi_1(p_1(\theta, \sigma(p_2|\theta)), p_2; \theta) d\sigma(p_2|\theta)$$

When firms offer substitutes, for any obedient $\sigma(p_2|\theta)$ we have that

$$\begin{aligned} \int_{p_2} q(p_1, p_2; \theta) d\sigma(p_2|\theta_H) &\geq \int_{p_2} q(p_1, p_2; \theta) d\sigma(p_2|\theta_L) \text{ and} \\ \int_{p_2} \frac{\partial q(p_1, p_2; \theta)}{\partial p_1} d\sigma(p_2|\theta_H) &\geq \int_{p_2} \frac{\partial q(p_1, p_2; \theta)}{\partial p_1} d\sigma(p_2|\theta_L) \text{ for all } p_1 \text{ and } \theta \end{aligned} \quad (13)$$

since $q(p_1, p_2; \theta)$ is strictly increasing in p_2 , $\int_0^x d\sigma(p_2|\theta_L) \geq \int_0^x d\sigma(p_2|\theta_H)$ for all x and $\frac{\partial^2 q(p_1, p_2; \theta)}{\partial p_1 \partial p_2} > 0$. Then, since $p_1(\theta, \sigma(p_2|\theta))$ is implicitly defined by

$$\int_{p_2} q(p_1, p_2; \theta) d\sigma(p_2|\theta) + p_1 \int_{p_2} \frac{\partial q(p_1, p_2; \theta)}{\partial p_1} d\sigma(p_2|\theta) = 0,$$

(13) implies that $p_1(\theta, \sigma(p_2|\theta_H)) \geq p_1(\theta, \sigma(p_2|\theta_L))$ for all $\theta \in \Theta$. Then, $\frac{\partial^2 q(p_1, p_2, \theta)}{\partial p_1 \partial p_2} \geq 0$ also implies that

$$\int_{p_2} q(p_1(\theta, \sigma(p_2|\theta_H)), p_2; \theta) d\sigma(p_2|\theta_H) \geq \int_{p_2} q(p_1(\theta, \sigma(p_2|\theta_L)), p_2; \theta) d\sigma(p_2|\theta_L).$$

Then, when firms offer substitutes,

$$\int_{p_2} \Pi_1(p_1(\theta, \sigma(p_2|\theta_H)), p_2; \theta) d\sigma(p_2|\theta_H) - \int_{p_2} \Pi_1(p_1(\theta, \sigma(p_2|\theta_L)), p_2; \theta) d\sigma(p_2|\theta_L) \geq 0 \quad (14)$$

for all $\theta \in \Theta$. By Leibnitz rule,

$$\begin{aligned}
& \frac{\partial \int_{p_2} \Pi_1(p_1(\theta, \sigma(p_2|\theta_H)), p_2; \theta) d\sigma(p_2|\theta_H)}{\partial \theta} \\
&= \int_{p_2} \frac{\partial p_1(t, \sigma(p_2|\theta_H))}{\partial t} \Big|_{t=\theta} \left[q(p_1(\theta, \sigma(p_2|\theta_H)), p_2; \theta) + p_1(\theta, \sigma(p_2|\theta_H)) \frac{\partial q(p_1, p_2; t)}{\partial p_1} \right] d\sigma(p_2|\theta_H) \\
&+ \int_{p_2} p_1(\theta, \sigma(p_2|\theta_H)) \frac{\partial q(p_1(\theta, \sigma(p_2|\theta_H)), p_2; t)}{\partial t} \Big|_{t=\theta} d\sigma(p_2|\theta_H) \\
&= \int_{p_2} p_1(\theta, \sigma(p_2|\theta_H)) \frac{\partial q(p_1(\theta, \sigma(p_2|\theta_H)), p_2; t)}{\partial t} \Big|_{t=\theta} d\sigma(p_2|\theta_H)
\end{aligned}$$

where the last inequality holds by the first order condition of the informed firm's pricing decision. Similarly,

$$\frac{\partial \int_{p_2} \Pi_1(p_1(\theta, \sigma(p_2|\theta_L)), p_2; \theta) d\sigma(p_2|\theta_L)}{\partial \theta} = \int_{p_2} p_1(\theta, \sigma(p_2|\theta_L)) \frac{\partial q(p_1(\theta, \sigma(p_2|\theta_L)), p_2; t)}{\partial t} \Big|_{t=\theta} d\sigma(p_2|\theta_L)$$

Then, the left-hand side of (14) is non-decreasing in θ since

$$\int_{p_2} p_1(\theta, \sigma(p_2|\theta_H)) \frac{\partial q(p_1(\theta, \sigma(p_2|\theta_H)), p_2; t)}{\partial t} \Big|_{t=\theta} d\sigma(p_2|\theta_H) \geq \int_{p_2} p_1(\theta, \sigma(p_2|\theta_L)) \frac{\partial q(p_1(\theta, \sigma(p_2|\theta_L)), p_2; t)}{\partial t} \Big|_{t=\theta} d\sigma(p_2|\theta_L)$$

because $p_1(\theta, \sigma(p_2|\theta_H)) > p_1(\theta, \sigma(p_2|\theta_L))$ and $\frac{\partial^2 q(p_1, p_2; \theta)}{\partial \theta \partial p_2} > 0$. Thus, when firms offer substitutes,

$$\begin{aligned}
& \int_{p_2} \Pi_1(p_1(\theta_H, \sigma(p_2|\theta_H)), p_2; \theta_H) d\sigma(p_2|\theta_H) - \int_{p_2} \Pi_1(p_1(\theta_H, \sigma(p_2|\theta_L)), p_2; \theta_H) d\sigma(p_2|\theta_L) \\
& \geq \int_{p_2} \Pi_1(p_1(\theta_L, \sigma(p_2|\theta_H)), p_2; \theta_L) d\sigma(p_2|\theta_H) - \int_{p_2} \Pi_1(p_1(\theta_L, \sigma(p_2|\theta_L)), p_2; \theta_L) d\sigma(p_2|\theta_L)
\end{aligned}$$

which implies that $\mathbb{E}_\sigma[\Pi_1^*(p_1, p_2; \theta)|\theta]$ is supermodular in θ and p_2 . The proof for the complement case is analogous. ■

Proof. Proposition 6. Consider first the case in which $\mathbb{E}_\sigma[\Pi_1^*(p_1, p_2; \theta)|\theta]$ is supermodular in p_2 and θ . Next, I show that for all σ and $p_2 \in [p^F(\theta_L), p^F(\theta_H)]$,

$$\mathbb{E}_{\sigma^F}[\Pi_1^*(p_1, p_2; \theta_L)|\theta_L] \leq \mathbb{E}_\sigma[\Pi_1^*(p_1, p_2; \theta_L)|\theta_L] \leq \mathbb{E}_\sigma[\Pi_1^*(p_1, p_2; \theta_H)|\theta_H] \leq \mathbb{E}_{\sigma^F}[\Pi_1^*(p_1, p_2; \theta_H)|\theta_H].$$

That is,

$$\begin{aligned}
& \Pi_1(p_1(\theta_L, \sigma^F(p_2|\theta_L)), p^F(\theta_L); \theta_L) \leq \int_{p_2} \Pi_1(p_1(\theta_L, \sigma(p_2|\theta_L)), p_2; \theta_L) d\sigma(p_2|\theta_L) \\
& \leq \int_{p_2} \Pi_1(p_1(\theta_H, \sigma(p_2|\theta_H)), p_2; \theta_H) d\sigma(p_2|\theta_H) \leq \Pi_1(p_1(\theta_H, \sigma^F(p_2|\theta_H)), p^F(\theta_H); \theta_H)
\end{aligned}$$

First,

$$\begin{aligned} \int_{p_2} \Pi_1(p_1(\theta_L, \sigma(p_2|\theta_L)), p_2; \theta_L) d\sigma(p_2|\theta_L) &\geq \int_{p_2} \Pi_1(p_1(\theta_L, \sigma^F(p_2|\theta_L)), p_2; \theta_L) d\sigma^F(p_2|\theta_L) \\ &= \Pi_1(p_1(\theta_L, \sigma^F(p_2|\theta_L)), p^F(\theta_L); \theta_L) \end{aligned} \quad (15)$$

since $\sigma^F(p_2|\theta_L)$ recommends $p^F(\theta_L)$ with probability 1, the informed firm's demand is increasing in p_2 and $p_1(\theta_L, \sigma(p_2|\theta_L)) \geq p_1(\theta_L, \sigma^F(p_2|\theta_L))$.²⁸ Similarly,

$$\begin{aligned} \int_{p_2} \Pi_1(p_1(\theta_H, \sigma(p_2|\theta_H)), p_2; \theta_H) d\sigma(p_2|\theta_H) &\leq \int_{p_2} \Pi_1(p_1(\theta_H, \sigma^F(p_2|\theta_H)), p_2; \theta_H) d\sigma^F(p_2|\theta_H) \\ &= \Pi_1(p_1(\theta_H, \sigma^F(p_2|\theta_H)), p^F(\theta_H); \theta_H) \end{aligned} \quad (16)$$

because $\sigma^F(p_2|\theta_H)$ recommends $p^F(\theta_H)$ with probability 1, the informed firm's demand is increasing in p_2 and $p_1(\theta_H, \sigma(p_2|\theta_H)) \leq p_1(\theta_H, \sigma^F(p_2|\theta_H))$.

Second, supermodularity implies that

$$\int_{p_2} \Pi_1(p_1(\theta_L, \sigma(p_2|\theta_L)), p_2; \theta_L) d\sigma(p_2|\theta_L) \leq \int_{p_2} \Pi_1(p_1(\theta_H, \sigma(p_2|\theta_H)), p_2; \theta_H) d\sigma(p_2|\theta_H) \quad (17)$$

since $p_1(\theta_L, \sigma(p_2|\theta_L)) \leq p_1(\theta_H, \sigma(p_2|\theta_H))$, $\frac{\partial^2 q(p_1, p_2; \theta)}{\partial \theta \partial p_2} > 0$ and the state is a positive demand shifter, implying that $\sigma(p_2|\theta_H)$ recommends on average higher prices than $\sigma(p_2|\theta_L)$. Thus, (15), (16) and (17) imply that

$$\begin{aligned} \Pi_1(p_1(\theta_L, \sigma^F(p_2|\theta_L)), p^F(\theta_L); \theta_L) &\leq \int_{p_2} \Pi_1(p_1(\theta_L, \sigma(p_2|\theta_L)), p_2; \theta_L) d\sigma(p_2|\theta_L) \\ &\leq \int_{p_2} \Pi_1(p_1(\theta_H, \sigma(p_2|\theta_H)), p_2; \theta_H) d\sigma(p_2|\theta_H) \leq \Pi_1(p_1(\theta_H, \sigma^F(p_2|\theta_H)), p^F(\theta_H); \theta_H) \end{aligned}$$

Then,

$$\begin{aligned} \mathbb{E}_{\sigma^F, \mu}[\Pi_1^*(p_1, p_2; \theta)] &= \sum_{\theta \in \Theta} \mu_\theta \Pi_1(p_1(\theta, \sigma^F(p_2|\theta)), p^F(\theta); \theta) \\ &\geq \sum_{\theta \in \Theta} \mu_\theta \int_{p_2} \Pi_1(p_1(\theta, \sigma(p_2|\theta)), p_2; \theta) d\sigma(p_2|\theta) \\ &= \mathbb{E}_{\sigma, \mu}[\Pi_1^*(p_1, p_2; \theta)] \end{aligned}$$

where the inequality holds by Jensen's inequality.

²⁸The proof of $p_1(\theta_L, \sigma(p_2|\theta_L)) \geq p_1(\theta_L, \sigma^F(p_2|\theta_L))$ follows an analogous argument as in Lemma 18.

Consider now the case in which $\mathbb{E}_\sigma[\Pi_1^*(p_1, p_2; \theta)|\theta]$ is submodular in θ and $\sigma(p_2|\theta)$. Analogously as in the supermodular case, it is possible to show that

$$\begin{aligned} \Pi_1(p_1(\theta_L, \sigma^N(p_2|\theta_L)), p^N; \theta_L) &\leq \int_{p_2} \Pi_1(p_1(\theta_L, \sigma(p_2|\theta_L)), p_2; \theta_L) d\sigma(p_2|\theta_L) \\ &\leq \int_{p_2} \Pi_1(p_1(\theta_H, \sigma(p_2|\theta_H)), p_2; \theta_H) d\sigma(p_2|\theta_H) \leq \Pi_1(p_1(\theta_H, \sigma^N(p_2|\theta_H)), p^N; \theta_H) \end{aligned}$$

which in turn implies that

$$\begin{aligned} \mathbb{E}_{\sigma^N, \mu}[\Pi_1^*(p_1, p_2; \theta)] &= \sum_{\theta \in \Theta} \mu_\theta \Pi_1(p_1(\theta, p^N), p^N; \theta) \\ &\geq \sum_{\theta \in \Theta} \mu_\theta \int_{p_2} \Pi_1(p_1(\theta, \sigma(p_2|\theta)), p_2; \theta) d\sigma(p_2|\theta_H) \\ &= \mathbb{E}_{\sigma, \mu}[\Pi_1^*(p_1, p_2; \theta)] \end{aligned}$$

where the inequality holds by Jensen's inequality. ■

A.4 Extensions proofs

Proof. Lemma 6. Proposition 1 and Lemma 5 imply that full disclosure is optimal if firms offer imperfect substitutes. If firms offer complements, the informed firm prefers to not disclose her private information whereas the uninformed firm prefers to learn the state. First, full disclosure yields higher producer surplus than no disclosure if and only if $\delta < \frac{2}{1+\sqrt{2}}$. Next, I show that full disclosure is optimal when firms offer complements if $\delta < \frac{2}{1+\sqrt{2}}$ and no disclosure is optimal otherwise.

Consider first the case in which $\delta < \frac{2}{1+\sqrt{2}}$. The difference in expected producer surplus of full disclosure σ^F and disclosure policy σ is

$$PS(\sigma^F) - PS(\sigma) \geq \left(a + b - \frac{b^2}{4a}\right) \mathbb{E}_\mu \left[\left(\frac{\theta}{2a-b} - \mathbb{E}_\sigma[p_2|\theta] \right)^2 \right] \geq 0$$

The first inequality holds by Jensen's inequality, $a > |b|$ and $b < 0$, whereas the second one holds for all $\delta < \frac{2}{1+\sqrt{2}}$. Hence, full disclosure is optimal if $\delta < \frac{2}{1+\sqrt{2}}$.

Consider now the case in which $\delta \geq \frac{2}{1+\sqrt{2}}$. The difference in expected producer surplus of no disclosure σ^N and disclosure policy σ is

$$PS(\sigma^N) - PS(\sigma) = a\mathbb{V}_{(\mu, \sigma)}[p_2] - (1 - \delta) \text{Cov}_{(\mu, \sigma)}[p_2, \theta] + \frac{3b}{4} \cdot \delta \mathbb{V}_\mu[\mathbb{E}_\sigma[p_2|\theta]]$$

Note that this difference is a strictly increasing function of δ because

$$\frac{\partial PS(\sigma^N) - PS(\sigma)}{\partial \delta} = \text{Cov}_{(\mu, \sigma)}[p_2, \theta] + \frac{3b}{4} \mathbb{V}_\mu[\mathbb{E}_\sigma[p_2|\theta]],$$

$$\text{Cov}_{(\mu, \sigma)}[p_2, \theta] > 2a\mathbb{V}_{(\mu, \sigma)}[p_2], \quad \delta < 1 \quad \text{and} \quad \mathbb{V}_{(\mu, \sigma)}[p_2] \geq \mathbb{V}_\mu[\mathbb{E}_\sigma[p_2|\theta]] \geq 0.$$

This implies that

$$PS(\sigma^N) - PS(\sigma) \geq a\mathbb{V}_{(\mu,\sigma)}[p_2] - \left(1 - \frac{2}{1 + \sqrt{2}}\right) Cov_{(\mu,\sigma)}[p_2, \theta] + \frac{3b}{4} \cdot \frac{2}{1 + \sqrt{2}} \mathbb{V}_\mu[\mathbb{E}_\sigma[p_2|\theta]] \geq 0$$

for all $\delta > \frac{2}{1 + \sqrt{2}}$ since $Cov_{(\mu,\sigma)}[p_2, \theta] \in [2a\mathbb{V}_{(\mu,\sigma)}[p_2], (2a - b)\mathbb{V}_{(\mu,\sigma)}[p_2]]$ and $\mathbb{V}_{(\mu,\sigma)}[p_2] \geq \mathbb{V}_\mu[\mathbb{E}_\sigma[p_2|\theta]]$.²⁹ Hence, no disclosure is optimal if $\delta > \frac{2}{1 + \sqrt{2}}$. ■

Proof. Lemma 7. With public signals, the informed firm's expected equilibrium profits, given by

$$\mathbb{E}_{(\mu,\sigma^{Pub})}[\Pi_1^*((p_1, p_2); \theta)] = a\mathbb{E}_\mu \left[\mathbb{E}_{\sigma^{Pub}} \left[\left(\frac{\theta + bp_2}{2a} \right)^2 \middle| \theta \right] \right],$$

are higher than its expected equilibrium profits with private signals. This holds because

$$\begin{aligned} \mathbb{E}_{(\mu,\sigma^{Pub})}[\Pi_1^*((p_1, p_2); \theta)] &= a\mathbb{E}_\mu \left[\mathbb{E}_{\sigma^{Pub}} \left[\left(\frac{\theta + bp_2}{2a} \right)^2 \middle| \theta \right] \right] \geq a\mathbb{E}_\mu \left[\left(\mathbb{E}_{\sigma^{Pub}} \left[\frac{\theta + bp_2}{2a} \middle| \theta \right] \right)^2 \right] \\ &= a\mathbb{E}_\mu \left[\left(\frac{\theta + b\mathbb{E}_{\sigma^{Pub}}[p_2|\theta]}{2a} \right)^2 \right] = a\mathbb{E}_\mu \left[\left(\frac{\theta + b\mathbb{E}_{\sigma^{Priv}}[p_2|\theta]}{2a} \right)^2 \right] \\ &= \mathbb{E}_{(\mu,\sigma^{Priv})}[\Pi_1^*((p_1, p_2); \theta)] \end{aligned}$$

where the inequality holds by Jensen's inequality. ■

Proof. Lemma 8. The expected consumer surplus with public disclosure is given by

$$CS(\sigma^{Pub}) = \frac{1}{2a} \mathbb{E}_\mu \left[\mathbb{E}_{\sigma^{Pub}} \left[q_1 \left(\frac{\theta + bp_2}{2a}, p_2; \theta \right)^2 + q_2 \left(p_2, \frac{\theta + bp_2}{2a}; \theta \right)^2 \middle| \theta \right] \right],$$

whereas expected consumer surplus with private disclosure is

$$CS(\sigma^{Priv}) = \frac{1}{2a} \mathbb{E}_\mu \left[\mathbb{E}_{\sigma^{Priv}} \left[q_1 \left(\frac{\theta + b\mathbb{E}_{\sigma^{Priv}}[p_2|\theta]}{2a}, p_2; \theta \right)^2 + q_2 \left(p_2, \frac{\theta + b\mathbb{E}_{\sigma^{Priv}}[p_2|\theta]}{2a}; \theta \right)^2 \middle| \theta \right] \right].$$

The difference between expected consumer surplus with private and public disclosure is

$$CS(\sigma^{Priv}) - CS(\sigma^{Pub}) = \frac{b^2}{8a} \left(7 - \frac{b^2}{a^2} \right) (\mathbb{E}_{(\mu,\sigma)}[p_2^2] - \mathbb{E}_\mu [\mathbb{E}_\sigma[p_2|\theta]^2])$$

Then, $CS(\sigma^{Priv}) \geq CS(\sigma^{Pub})$ because $a > |b|$ and

$$\mathbb{E}_{(\mu,\sigma)}[p_2^2] - \mathbb{E}_\mu [\mathbb{E}_\sigma[p_2|\theta]^2] = \mathbb{E}_\mu[\mathbb{E}_\sigma[p_2^2|\theta]] - \mathbb{E}_\mu [\mathbb{E}_\sigma[p_2|\theta]^2] \geq 0$$

²⁹The highest covariance between prices and the state occurs with full disclosure. In this case, $p_2(\theta) = \frac{\theta}{2a-b}$. Hence, $Cov_{(\mu,\sigma)}[p_2, \theta] \leq Cov_{(\mu,\sigma)}[p_2, (2a - b)p_2] = (2a - b)\mathbb{V}_{(\mu,\sigma)}[p_2]$.

where the equality holds by the law of iterated expectations and the inequality by Jensen's inequality. ■

Proof. Lemma 9. First, no disclosure is optimal when firms offer complements since $CS(\sigma^{Pub}) \leq CS(\sigma^{Priv})$. Similarly, no disclosure is optimal when firms offer substitutes and firms offer sufficiently far substitutes ($\delta < \hat{c}$). Consider then the case in which firms offer substitutes ($b > 0$) and $\delta \geq \hat{c}$. The expected gain of consumer surplus with public disclosure with respect to no disclosure is given by:

$$CS(\sigma^{Pub}) - CS(\sigma^N)$$

where the first inequality holds by definition of variance, covariance and δ . The second inequality holds because $CS(\sigma^{Pub}) < CS(\sigma^N)$ for $\delta = 0$. ■

Proof. Lemma 10. The designer commits to an information structure with private signals, denoted by $\hat{\psi}_k$, to share all the informed firm's private information with k firms and share no information with $N - 1 - k$ firms, where $k \in \{0, 1, 2, \dots, N - 1\}$. Firms who observe a perfectly informative signal condition their pricing choices on the state and select a mapping $p^F : \Theta \rightarrow \mathbb{R}_+$ to maximize their expected profits, whereas firms who observe no information select a price $p^N \in \mathbb{R}_+$ to maximize their expected profits. Equilibrium prices are

$$p^F(\theta_L) = \frac{\theta_L(2a(N-1) - bk) + b\mu_H(N-k-1)(\theta_H - \theta_L)}{(2a-b)(2a(N-1) - bk)},$$

$$p^F(\theta_H) = \frac{\theta_H(2a(N-1) - bk) - b\mu_L(N-k-1)(\theta_H - \theta_L)}{(2a-b)(2a(N-1) - bk)}, \text{ and } p^N = \frac{\mu_L\theta_L + \mu_H\theta_H}{2a-b}.$$

Consider first the case in which the designer's objective is to maximize the informed firm's expected equilibrium profits, given by

$$\mathbb{E}[\Pi_1^*(\hat{\psi}_k)] = a \sum_{\theta \in \Theta} \mu_\theta p^F(\theta)^2.$$

The informed firm's expected equilibrium profits are maximized by sharing its private information with all other firms ($k^* = N - 1$). Similarly, when the designer's objective is to maximize expected producer surplus, given by

$$PS(\hat{\psi}_k) = (N - k - 1)a(p^N)^2 + (k + 1)a \sum_{\theta \in \Theta} \mu_\theta p^F(\theta)^2,$$

it is optimal to share information with all firms ($k^* = N - 1$), eliminating all information asymmetry between firms.

In contrast, if the designer's objective is to maximize expected consumer surplus, information disclosure between firms is at least partially restricted. Expected consumer surplus, given by,

$$CS(\hat{\psi}_k) = \frac{(k+1)}{2a} \sum_{\theta \in \Theta} \mu_\theta \left[\theta + b \frac{(N-k-1)}{N-1} p^N - \left(a - b \frac{k}{N-1} \right) p^F(\theta) \right]^2 \\ + \frac{(N-k-1)}{2a} \sum_{\theta \in \Theta} \mu_\theta \left[\theta + b \frac{k+1}{N-1} p^F(\theta) - \left(a - b \frac{N-k-2}{N-1} \right) p^N \right]^2$$

The optimal information structure, characterized by $k^*(N, \delta)$, is determined by the degree of substitution and the number of firms in the market, where

$$k^*(N, \delta) = \begin{cases} 0 & \text{if } \delta \leq \frac{3}{4} \text{ for all } N \geq 3 \\ 0 & \text{if } \delta \in \left(\frac{3}{4}, 0.76 \right) \text{ and } N \in \left[3, 1 + \frac{1}{2} \sqrt{\frac{\delta^2}{4\delta-3}} - \frac{\delta}{2} \right] \\ f(N, \delta) & \text{otherwise} \end{cases}$$

and

$$f(N, \delta) = \lceil \frac{2(N-1)(\delta^3 + \delta^2(4N-5) + \delta(4N-7)(N-1) - 3(N-1)^2)}{\delta(\delta+(N-1))(\delta+3(N-1))} \rceil \text{ if} \\ CS \left[\hat{\pi} \lceil \frac{2(N-1)(\delta^3 + \delta^2(4N-5) + \delta(4N-7)(N-1) - 3(N-1)^2)}{\delta(\delta+(N-1))(\delta+3(N-1))} \rceil \right] \geq CS \left[\hat{\pi} \lfloor \frac{2(N-1)(\delta^3 + \delta^2(4N-5) + \delta(4N-7)(N-1) - 3(N-1)^2)}{\delta(\delta+(N-1))(\delta+3(N-1))} \rfloor \right] \text{ and} \\ f(N, \delta) = \lfloor \frac{2(N-1)(\delta^3 + \delta^2(4N-5) + \delta(4N-7)(N-1) - 3(N-1)^2)}{\delta(\delta+(N-1))(\delta+3(N-1))} \rfloor \text{ otherwise.}$$

■